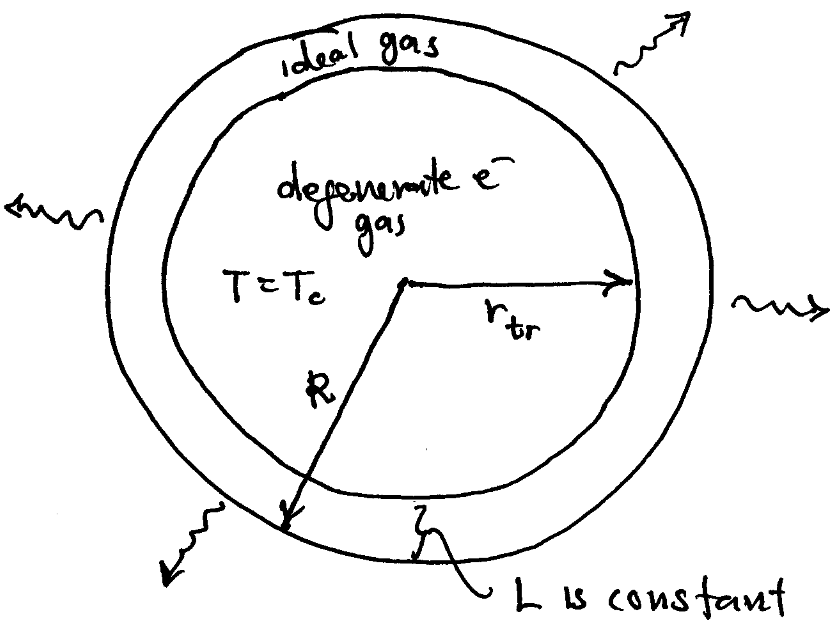
1.

WD cooling

assumptions:

- core is degenerate, contains nearly all M, R
- thin radiative envelope surrounds the core (ideal gas)
- transition b/w core and envelope is abrupt, at $r = r_{tr}$



- conduction dominates in core — nearly isothermal
- $T_{tr} = T_{core}$
- no ν losses, ϵ_{nuc}, or contraction

recall the radiative envelope discussion when we covered polytropes

completely radiative: $\nabla = \nabla_{rad} = \dfrac{d\log T}{d\log P} = \dfrac{3}{16\pi a c G} \dfrac{P\bar{\kappa}}{T^4} \underbrace{\dfrac{L_*}{M_*}}_{\text{thin envelope}}$

we took: P as ideal gas

$$\kappa = \kappa_0 \rho^{\nu} T^{-s} = \kappa_g P^{\nu} T^{-\nu-s}$$

$$\kappa_g = \kappa_0 \left(\frac{\mu m_u}{k}\right)^{\nu}$$

We rewrote this as

$$T^4 \, d\log T = \left(\frac{3\,\bar{\kappa}}{16\pi a c G}\right) P \, d\log P$$

or

$$T^{\nu+s+3} \, dT = \left(\frac{3\,\kappa_g L_*}{16\pi a c G M_*}\right) P^{\nu} \, dP$$

and we integrated from T_0, P_0 to $T(r) \geq T_0$, $P(r) \geq P_0$
(inward)

giving

$$P^{\nu+1}\left[1-\left(\frac{P_0}{P}\right)^{\nu+1}\right] = \frac{16\pi a c G}{3\kappa_g}\,\frac{M_*}{L_*}\,\frac{\nu+1}{\nu+s+4}\,T^{\nu+s+4}\left[1-\left(\frac{T_0}{T}\right)^{\nu+s+4}\right]$$

w/ $\nu+s+4 \neq 0$

Now, if $\nu+s+4 > 0$ and $\nu+1 > 0$, then at a depth beneath the surface, $P(r) \gg P_0$, $T(r) \gg T_0$, giving

$$P^{\nu+1} = \frac{16\pi a c G}{3\kappa_g}\,\frac{M_*}{L_*}\,\frac{\nu+1}{\nu+s+4}\,T^{\nu+s+4}$$

this has the form

$$P = K' \, T^{1+n_{\text{eff}}} \qquad \text{w/} \quad n_{\text{eff}} = \frac{s+3}{\nu+1}$$

and

$$K' = \left[\frac{1}{1+n_{\text{eff}}}\,\frac{16\pi a c G M_*}{3\underline{\kappa_g} L_*}\left(\frac{k}{\mu m_u}\right)^{\nu}\right]^{1/(\nu+1)}$$

κ_0 not κ_g

If we have a constant composition then $K' =$ constant

and

$$P = \frac{\rho k T}{\mu m_0} = K' T^{1+n_{eff}} \longrightarrow \rho \sim T^{n_{eff}}$$

$$\therefore \; T \sim \rho^{1/n_{eff}} \quad \text{and} \quad P \sim \rho T \sim \rho^{1+1/n_{eff}} \quad \text{(a polytrope!)}$$

4.

What is the transition from core to envelope?

$$E_f \sim kT$$

$$\frac{p_f^2}{2m_e} \sim kT$$

or using $x_F = \frac{p_F}{m_e c}$

$$x_F^2 \sim \frac{2kT}{m_e c^2}$$

from our number density constraint, we know

$$n_e = \frac{8\pi}{3}\left(\frac{m_e c}{h}\right)^3 x_F^3 \qquad \text{(HKT 3.50)}$$

$$\therefore \quad n_e \sim \left(\frac{2kT}{m_e c^2}\right)^{3/2} \frac{8\pi}{3}\left(\frac{m_e c}{h}\right)^3$$

$$\frac{\rho}{\mu_e m_u} \sim \left(\frac{2kT}{m_e c^2}\right)^{3/2} \frac{8\pi}{3}\left(\frac{m_e c}{h}\right)^3$$

$$\left(\frac{\rho}{\mu_e m_u}\right)^{2/3} \sim \frac{2kT}{m_e \cancel{c^2}}\left(\frac{8\pi}{3}\right)^{2/3}\left(\frac{m_e \cancel{c}}{h}\right)^2$$

or

$$kT \sim \left(\frac{\rho}{\mu_e}\right)^{2/3} \frac{1}{2m_e}\left(\frac{3h^3}{8\pi}\right)^{2/3} \frac{1}{m_u^{2/3}}$$

this is

$$T \sim 2.9\times10^5 \left(\rho/\mu_e\right)^{2/3} \qquad [CGS]$$

or

$$\left(\frac{\rho}{\mu_e}\right) \sim \underbrace{6\times10^{-9}}_{\equiv \alpha} T^{3/2} \qquad [CGS]$$

The transition occurs when

$T_{tr} \sim T_c$ (since conduction is so efficient)

requiring that the pressure is continuous across the transition,

$$P_{tr} = K' T_{tr}^{1+n_{eff}} = \frac{\rho_{tr} k T_{tr}}{\mu m_u} = \alpha \mu_e \frac{k}{\mu m_u} T_{tr}^{1+3/2}$$

use $\rho_{tr} \sim \alpha \mu_e T_{tr}^{3/2}$

We'll assume that the envelope composition is the same as the core (probably not true)

K' depends on L_*, M_*, μ, and κ

Bound-free opacity gives

$$\kappa_{bf} \sim 4 \times 10^{25} \rho T^{-7/2} \text{ cm}^2 \text{ g}^{-1}$$

so $\nu = 1$, $s = +7/2$, and $n_{eff} = \frac{s+3}{\nu+1} = 3.25$

$$K' = \left[\frac{1}{1+n_{eff}} \frac{16\pi a c G M_*}{3 \kappa_0 L_*} \left(\frac{k}{\mu m_u}\right)\right]^{1/2}$$

$$= 8 \times 10^{-15} \mu^{-1/2} \left(\frac{M_*}{M_\odot}\right)^{1/2} \left(\frac{L_*}{L_\odot}\right)^{-1/2} \quad [CGS]$$

taking $\mu_e = 2$, we have

$$K' T_{tr}^{1+3.25} = \alpha (2) \frac{k}{\mu m_u} T_{tr}^{1+3/2}$$

$$8 \times 10^{-15} \mu^{-1/2} \left(\frac{M_*}{M_\odot}\right)^{1/2} \left(\frac{L_*}{L_\odot}\right)^{-1/2} T_{tr}^{17/4} = 6 \times 10^{-9} \cdot 2 \cdot \frac{k}{\mu m_u} T_{tr}^{5/2}$$

Together this gives

$$\frac{L_*}{L_\odot} = 6.5\times10^{-29}\,\mu\left(\frac{M_*}{M_\odot}\right)T_{tr}^{7/2} \qquad [CGS]$$

Note: we can get the thickness of the envelope using the polytrope structure we derived earlier

Assume no internal heat sources, then $R \sim$ const

most of the heat is in the ions:

$$c_v = \left.\frac{\partial e}{\partial T}\right|_\rho = \frac{3}{2}\frac{k}{\mu_I m_u} \sim \frac{1.2\times10^8}{\mu_I}\ \text{erg/g/K}$$

the luminosity is just

$$L_* = -\frac{dE_{ion}}{dt} = -\underbrace{c_v M_*}_{\substack{\text{core has all the}\\ \text{mass and const } T}}\frac{dT_{tr}}{dt}$$

equating, starting w/ previous result

$$L_* = 6.5\times10^{-29}\,\mu\left(\frac{M_*}{M_\odot}\right)L_\odot T_{tr}^{7/2}$$

$$\frac{dL_*}{dt} = \frac{7}{2}\eta\,\mu\,M_* T_{tr}^{5/2}\frac{dT_{tr}}{dt} \qquad \left(\eta \equiv 6.5\times10^{-29}\frac{L_\odot}{M_\odot}\right)$$

but $T_{tr} = \left(\dfrac{L_*}{\eta\mu M_*}\right)^{2/7}$, so

$$\frac{dL_*}{dt} = \frac{7}{2}(\eta\mu)^{2/7}M_*^{2/7}L_*^{5/7}\underbrace{\left[-\frac{L_*}{c_v M_*}\right]}_{\frac{dT_{tr}}{dt}\text{ from specific heat}}$$

Or

$$\frac{dL_*}{dt} = -\frac{7}{2}(\eta\mu)^{2/7} M_*^{-5/7} L_*^{5/7+1} c_v^{-1}$$

we can integrate this

$$\int_{L_0}^{L_*} L^{-(5/7+1)}\, dL = -\frac{7}{2}\frac{(\eta\mu)^{2/7}}{M_*^{5/7} c_v}\int_0^{t_{cool}} dt$$

$$-\frac{7}{5} L^{-5/7}\Big|_{L_0}^{L_*} = -\frac{7}{2}\frac{(\eta\mu)^{2/7}}{M_*^{5/2} c_v}\, t_{cool}$$

$$\therefore\ t_{cool} = \frac{2}{5} c_v \frac{M_*^{5/2}}{(\eta\mu)^{2/7}} L_\odot^{-5/7}\left[\left(\frac{L_*}{L_\odot}\right)^{-5/7} - \left(\frac{L_0}{L_\odot}\right)^{-5/7}\right]$$

putting in $\eta = 6.5\times10^{-29}\,\frac{L_\odot}{M_\odot}$ and $c_v = \frac{1.25\times10^{8}}{\mu_I}$ erg/g/K

we find

$$t_{cool} \sim 6\times10^{6}\ \underline{\underline{\text{yr}}}\ \underbrace{\left(\frac{A}{12}\right)^{-1}}_{\mu_I = A}\left(\frac{M}{M_\odot}\right)^{5/7}\left(\frac{\mu}{2}\right)^{-2/7}\left[\left(\frac{L_*}{L_\odot}\right)^{-5/7} - \underbrace{\left(\frac{L_0}{L_\odot}\right)^{-5/7}}_{\text{negligible after long times}}\right]$$