Review of where we are:

Our equations of stellar structure are

$$\frac{dr}{dM} = \frac{1}{4\pi r^2 \rho}$$

$$\frac{dP}{dM} = -\frac{GM(r)}{4\pi r^4}$$ We just finished talking about $P(\rho, T)$

$$\frac{dL}{dM} = \epsilon$$

ϵ — we'll discuss this in Ch6

+ temperature evolution

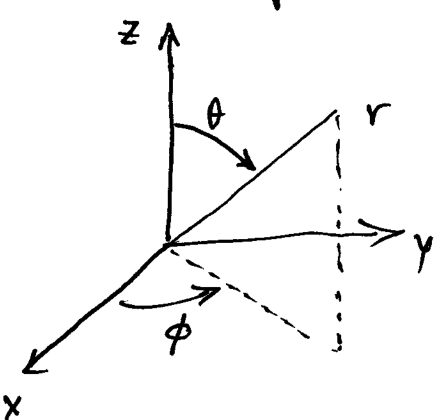
$$L = \frac{(4\pi r^2)^2 ac}{3\kappa}\frac{dT^4}{dM} \quad \text{(if radiation dominates)}$$

κ — we'll discuss this here

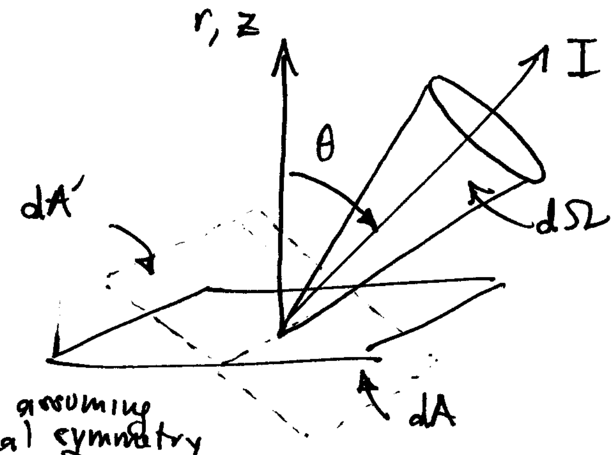
? if convection dominates (next chapter)

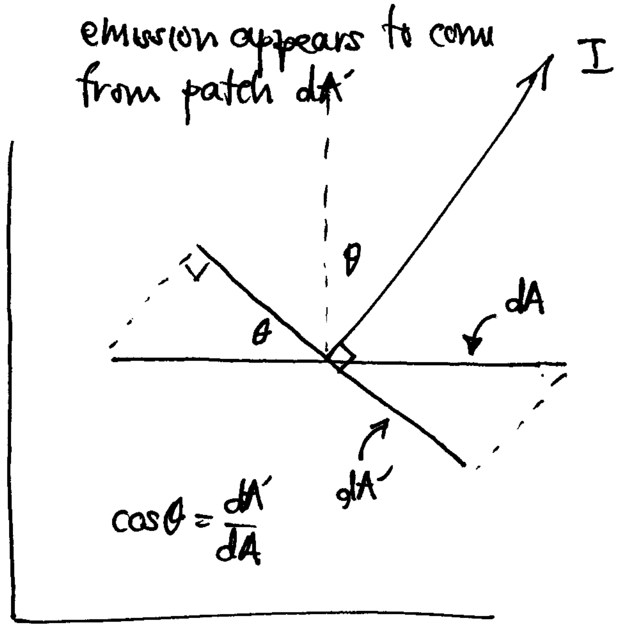
---

note our convention for spherical angles



2. We discussed (specific intensity when talking about blackbody radiation





We are assuming azimuthal symmetry

$$dE = I_\nu(\theta) \cos\theta \, dA \, d\nu \, d\Omega \, dt$$

We'll follow your book and start by considering all photons to have the same frequency

Then $dF = \frac{dE}{dA\,dt} = I(\theta)\cos\theta \, d\Omega$ (neglecting $d\nu$ for now)

flux ($erg/cm^2/s$)

Here dF is the flux passing through $d\Omega$ at an angle θ from the surface normal

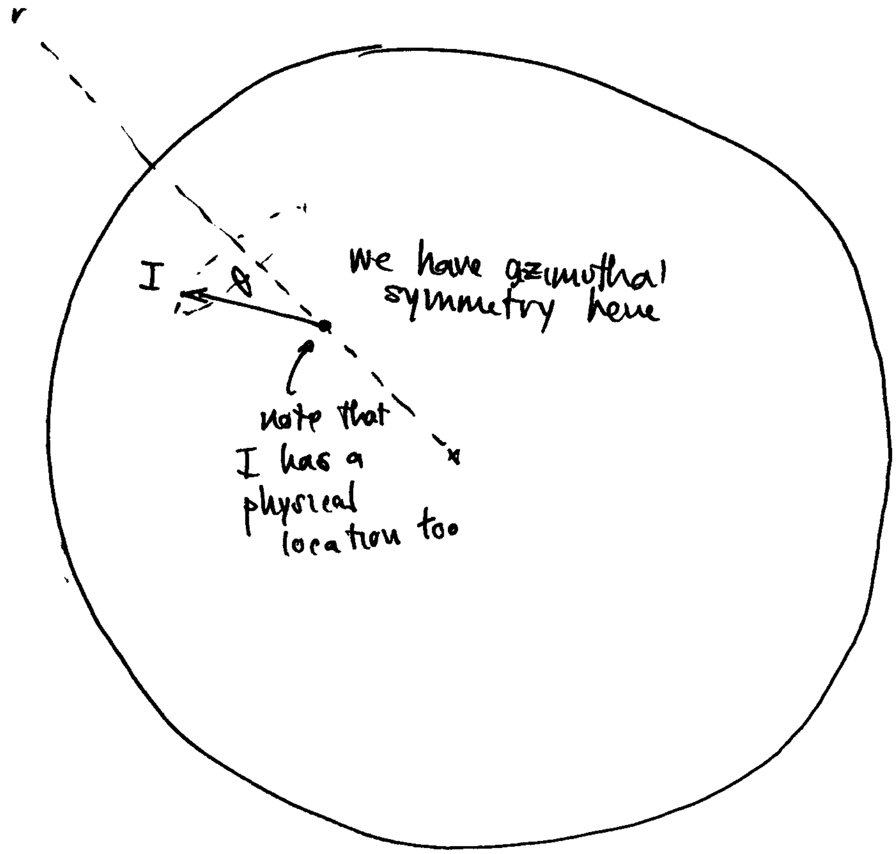
Note: • specific intensity has direction
• specific intensity does not fall off w/ distance (flux does) (assuming moving through vacuum)

specific intensity is <u>per unit solid angle</u>

It is a measure of surface brightness

2a. Why $I(\theta)$ and not $I(\theta, \phi)$?

We are assuming spherical symmetry (or later, plane-parallel) and θ is the angle from the outward radius vector (or vertical)



θ is basically telling us if the radiation is moving out of the star ($\theta = 0$) or inward ($\theta = \pi$)

---

surface brightness (from Choudhuri)

specific intensity is per unit solid angle — it measures surface brightness

As you move further away from a resolved object, both flux and angular size fall off as $1/r^2$

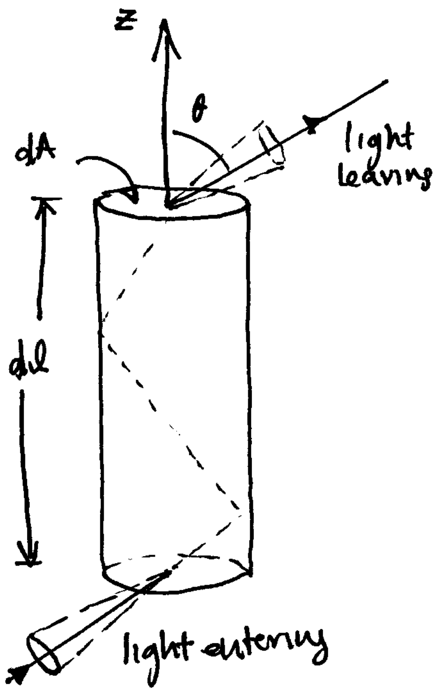
Surface brightness $\sim \frac{F}{\Omega} \sim I$ stays the same

Ex: look at streetlights at varying distances — the closer ones will be bigger in size, but the surface brightness at each will appear the same mean intensity

$$\langle I \rangle = \frac{1}{4\pi} \int I(\theta)\, d\Omega$$

If we are isotropic, then $I = \text{const}$, and $\langle I \rangle = I$
(note: blackbody radiation is isotropic)

How much energy is contained in the radiation field (see C&O)



Consider a trap, open at both ends. Light enters, bounces around, and exits the other end.

Radiation enters @ θ angle and travels at c for a time

$$dt = \frac{dl}{c\cos\theta}$$

∴ energy inside the trap (ignoring ν or λ dependence) is

$$dE = I\, dt\, dA \cos\theta\, d\Omega = I\, dA\, d\Omega\, \frac{dl}{c}$$

energy density (energy / volume) is

$$\frac{dE}{dV} = \frac{dE}{dA\, dl} = \frac{I}{c}\, d\Omega$$

to find the total energy density, integrate over all solid angles

$$U = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{I}{c}\, d\Omega = \frac{2\pi}{c} \int_{\theta=0}^{\pi} I \sin\theta\, d\theta$$

total energy density

4.

Note: if I is isotropic, then

$$U = \frac{4\pi}{c} I$$

It is common to express the angle θ as $\mu = \cos\theta$

Then $$U = \frac{2\pi}{c}\int_{1}^{-1} I(\mu)\,(-d\mu) = \frac{2\pi}{c}\int_{-1}^{1} I(\mu)\,d\mu$$

Now back to flux — what is the total flux at some point inside the star?

$$F = \int_{\Omega} I(\theta)\cos\theta\, d\Omega = 2\pi\int_{-1}^{1} I(\mu)\,\mu\, d\mu$$

note that if the radiation field is isotropic, then

$F = 0$!

(in your homework, you integrated over the outgoing Ω to get the surface flux from a black body)

No energy transport if I is isotropic

We are relying on photons to carry energy from the core to the surface, so we need an anisotropic I
— I must vary w/ θ or μ

(but LTE said that $\lambda_\gamma \ll H$, so we can only expect a very small anisotropy

4a

Recall that previously we had

$$p e_\gamma = 4\pi \int n(p)\, pc\, p^2\, dp = \frac{8\pi c}{h^3} \int_0^\infty \frac{p^3}{e^{pc/kT}-1}\, dp$$

for a photon, $E = h\nu = pc \longrightarrow p = \frac{h\nu}{c}$, $dp = \frac{h}{c} d\nu$

then

$$p e_\gamma = \frac{8\pi c}{h^3} \left(\frac{h}{c}\right)^4 \int_0^\infty \frac{\nu^3\, d\nu}{e^{h\nu/kT}-1} = U$$

this is what we are calling this now

$$\therefore\ u_\nu d\nu = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/kT}-1}\, d\nu \quad [\mathrm{erg\ cm^{-3}\ Hz^{-1}}\ \underbrace{\mathrm{Hz}}_{d\nu}]$$

$$\text{w/ } U = p e_\gamma = \int_0^\infty u_\nu\, d\nu = \underline{\underline{a T^4}}$$

result we found previously

Now Planck function is

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT}-1}$$

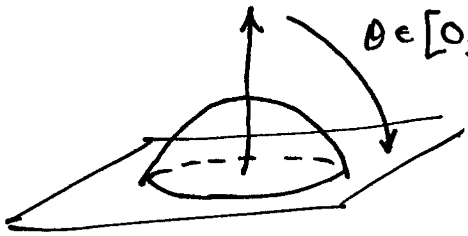
defining $B \equiv \int B_\nu\, d\nu$

$$\sigma \equiv \frac{ca}{4}$$

we have $B = I = \frac{c}{4\pi} U = \frac{ca}{4} \frac{T^4}{\pi} = \frac{\sigma T^4}{\pi}$

isotropic

Outward flux from surface of a blackbody



$$F = \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \; \underbrace{B}_{\text{integrated over } \nu} \cos\theta \; \underbrace{\sin\theta}_{\text{from } d\Omega}$$

$$= 2\pi B \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta$$

$\mu = \cos\theta$

$d\mu = -\sin\theta \, d\theta$

$$= 2\pi B \int_0^1 \mu \, d\mu = 2\pi B \left. \frac{\mu^2}{2} \right|_0^1 = \pi B = \sigma T^4 \; !$$

Sources (and sinks) of I

- $I(\theta)$ can increase by radiation scattering from another direction into θ
- direct emission from atoms at that location

$\Big\}\ j(\theta)$ — source (mass emission coefficient) (erg/g/s)

We express this as

$$dI = \overset{\text{addition}}{+}\, j(\theta)\,\rho\, ds$$

Sinks:
- scattering out of θ
- absorption

$\Big\}$ opacity, κ describes these

$$dI = -\underbrace{\kappa \rho\, I(\theta)}\, ds$$

note that ds is measured along the direction of I

note that the decrease depends on the amount of radiation we have

κ is in cm^2/g

Note: we are ignoring time-dependency in all of this!

Last time:

Our goal is to obtain the transport Eq. for radiation

(specific intensity is defined as

$$dE = I_\nu(\theta)\cos\theta \, dA \, d\nu \, d\Omega \, dt$$

moments:

$$\langle I \rangle = \frac{1}{4\pi}\int I(\theta)\, d\Omega \qquad \text{(mean intensity)}$$

$$U = \frac{1}{c}\int I(\theta)\, d\Omega \qquad \text{(energy density)}$$

$$= \frac{2\pi}{c}\int_{-1}^{1} I(\mu)\, d\mu$$

$$\mu \equiv \cos\theta$$

$$F = \int_\Omega I(\theta)\cos\theta \, d\Omega = 2\pi\int_{-1}^{1} I(\mu)\,\mu\, d\mu \qquad \text{(flux)}$$

Planck function

$$\underbrace{B}_{\text{integrated over } \nu} = \overbrace{I}^{\text{isotropic}} = \frac{c}{4\pi}U = \frac{\sigma T^4}{\pi}$$

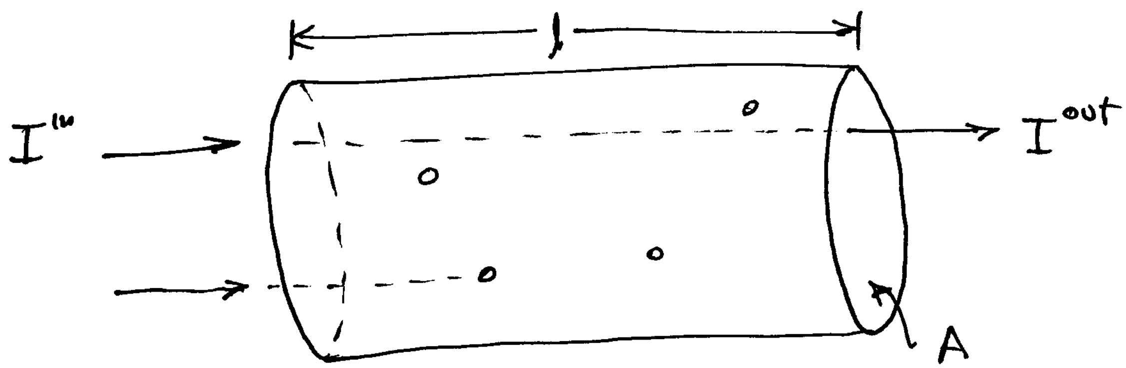
$$\underbrace{F(\mu>0)}_{\text{outward}} = \sigma T^4$$

radiation transfer $\quad \frac{dI(\theta)}{ds} = j(\theta)\rho - \kappa\rho I(\theta)$

B. Building some intuition about radiation & opacity (ignoring sources)

consider a cylinder w/ some absorbers, characterized by a size r_a, and a cross-section for absorption of $\sigma = \pi r_a^2$

Consider the case where the # density of absorbers, n, is small



if n is small, then we can assume that no two atoms lie on the same line of sight

Total absorption cross section is $\sigma_{tot} = N\sigma = nAl\sigma$

our assumption implies that $\sigma_{tot} \ll A$

—

Now, the fraction of incoming radiation that is absorbed is

$$f_{abs} = \frac{\sigma_{tot}}{A} = n\sigma l = \tau$$

τ optical depth

our assumption also means $\tau \ll 1$ (optically thin)

soon we'll see
opacity: $\tau = \kappa\rho l$ $\therefore$ $\kappa\rho = n\sigma$

7. Note that the mean free path is just

$$\lambda_\gamma = \frac{1}{n\sigma} = \frac{1}{\kappa\rho}$$

and then $\tau = \frac{l}{\lambda_\gamma}$ — optical depth is just the # of mean free paths we travel through

$\frac{1}{\kappa\rho}$ is the average distance a γ travels before being absorbed, so

$\kappa\rho$ is ~ the # of absorptions / unit length

Note if $\tau \gg 1$, then the above ideas are not necessarily true.

In particular τ is no longer the fraction of radiation absorbed.

We can recover the same behavior by dividing l into small slabs such that $d\tau = \kappa\rho \, dl \ll 1$, then

$$dI = -\kappa\rho \, dl \, I \quad \text{uses the fraction definition}$$

Integrating this, w/ constant κ, ρ, gives

$$I = I_0 e^{-\kappa\rho l} = I_0 e^{-\tau} \quad \text{(no sources!)}$$

$\uparrow$ incident radiation

Consider both absorption and emission

$$\boxed{\frac{dI(\theta)}{ds} = j(\theta)\rho - \kappa\rho I(\theta)}$$

— equation of radiation transfer. Note: for spherical there will be geometric terms

some notes (C&O):

if $dI/ds = 0$ (intensity does not vary w/ s)

then we have a balance of emission and absorption

rewriting:

$$-\frac{1}{\kappa\rho}\frac{dI}{ds} = I - \frac{j}{\kappa} = I - \underset{\sim}{S}$$

$$S \equiv j/\kappa$$

S is the source function, so in equilibrium $I = S$

if $dI/ds \neq 0$ then the balance will try to achieve $I = S$

Let's examine the tendency for I to approach S (C&O Ch 9)

our transfer equation is

$$-\frac{1}{\kappa\rho}\frac{dI}{ds} = I - S$$

We can see that if S is constant and κ and ρ are also constant, then

$$I(s) = I_0 e^{-\kappa\rho s} + S(1 - e^{-\kappa\rho s})$$

(proof by substitution:

$$\frac{dI}{ds} = -\kappa\rho I_0 e^{-\kappa\rho s} + \kappa\rho S e^{-\kappa\rho s}$$

then $$-\frac{1}{\kappa\rho}\frac{dI}{ds} = I_0 e^{-\kappa\rho s} - S e^{-\kappa\rho s} = I_0 e^{-\kappa\rho s} + S - S - S e^{-\kappa\rho s}$$

$$= I - S \;)$$

Now if we consider $S = 2I_0$ then the solution is

$$I(s) = I_0 e^{-\kappa\rho s} + 2I_0(1 - e^{-\kappa\rho s})$$

$$= I_0(2 - e^{-\kappa\rho s})$$

identifying mean free path as $\lambda_\gamma = \frac{1}{\kappa\rho}$, we have

$$I(s) = 2I_0\left(1 - \frac{1}{2}e^{-s/\lambda_\gamma}\right)$$

$\therefore$ after a few mean free paths, we have $I \sim 2I_0 = S$

9.

Consider I is isotropic, and spatially uniform, then

$$I = S \left(= \frac{j}{\kappa}\right) = \text{constant} \quad (\text{no transport})$$

the energy density in this case is

$$U = \frac{2\pi}{c}\int_{-1}^{1} I \, d\mu = \frac{4\pi}{c} I$$

but we also know that in LTE, $U = aT^4$ (last chapter)

$$\therefore \quad I = \frac{c}{4\pi} a T^4 = \frac{\sigma T^4}{\pi} = B(T)$$

frequency integrated Planck function

$$B(T) = \int_0^\infty B_\nu(T)\, d\nu$$

If we are <u>nearly</u> isotropic, then $I \sim B$, but how close?

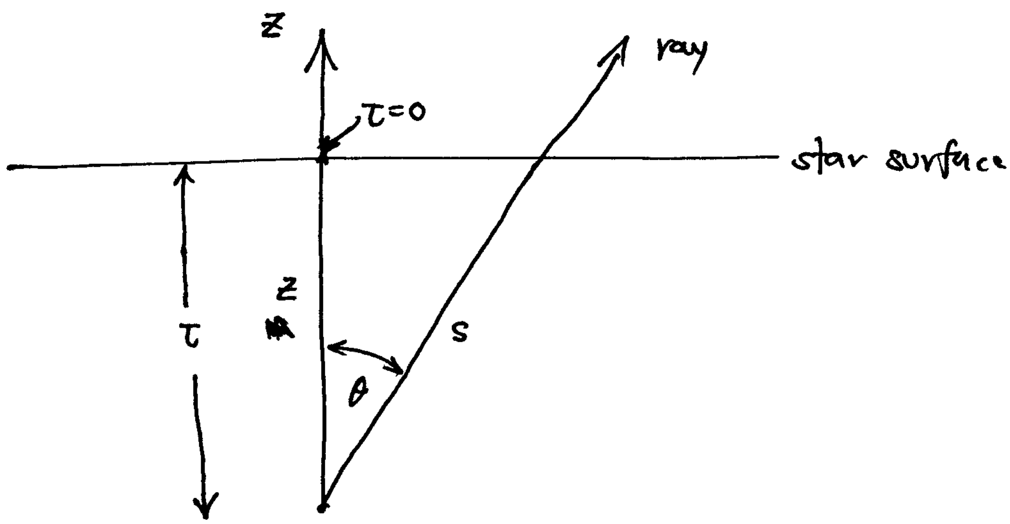
10. (see C&O §9)

Note that so far, we've been working w/ s as our coordinate — s is the distance measured along a ray

But for a star, z or r is what we want — so we need a measure that corresponds to a physical depth in the star.

(↑ plane-parallel)

As we'll see later, only at the surface/atmosphere do a lot of the complex radiation transport features come into play, so we'll consider a plane parallel geometry



We are going to work now w/ τ defined as the <u>vertical optical depth</u> — that is, this is directly vertical (or along some radius). In this case, it corresponds to a physical depth in the star

We see that $dz = \cos\theta \, ds$

so our equation of rad transfer becomes

$$-\frac{1}{\kappa\rho}\frac{dI_\nu}{ds} = I_\nu - S_\nu$$

$$-\cos\theta \frac{1}{\kappa_\nu\rho}\frac{dI_\nu}{dz} = I_\nu - S_\nu$$

recalling $\mu \equiv \cos\theta$

and $d\tau_\nu \equiv -\kappa_\nu \rho \, dz$ ← vertical position/depth in star

$$\boxed{\mu \frac{dI_\nu(\mu,\tau)}{d\tau_\nu} = I_\nu(\tau,\mu) - S_\nu(\tau,\mu)}$$

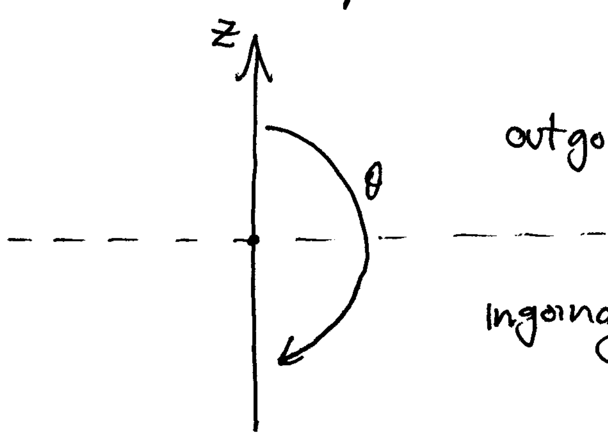
Let's now show that there is an I gradient — necessary for radiation transport

now note that

1. we can rewrite this as

$$\frac{d}{d\tau}\left[e^{-\tau/\mu} I\right] = -e^{-\tau/\mu}\frac{S}{\mu}$$

2. we can break μ up into outgoing and ingoing radiation



outgoing: $0 \leq \theta \leq \frac{\pi}{2} \rightarrow \mu = \cos\theta \geq 0$

ingoing: $\frac{\pi}{2} \leq \theta \leq \pi \rightarrow \mu = \cos\theta \leq 0$

We can integrate this new form easily

$$e^{-\tau/\mu} I \Big|_{\tau_0}^{\tau} = - \int_{\tau_0}^{\tau} e^{-t/\mu} \frac{S}{\mu}\, dt$$

τ_0: τ reference; t: integration variable

$$e^{-\tau/\mu} I(\tau,\mu) = e^{-\tau_0/\mu} I(\tau_0,\mu) - \int_{\tau_0}^{\tau} e^{-t/\mu} \frac{S}{\mu}\, dt$$

or finally

$$I(\tau,\mu) = e^{-(\tau_0-\tau)/\mu} I(\tau_0,\mu) - \int_{\tau_0}^{\tau} e^{-(t-\tau)/\mu} \frac{S}{\mu}\, dt$$

consider outgoing, $\mu \geq 0$

we want $\tau_0 \to \infty$ (our reference is deep in the star), then

$$I(\tau,\mu>0) = \int_{\tau}^{\infty} e^{-(t-\tau)/\mu} \frac{S(t)}{\mu}\, dt$$

now consider inward. Take $\tau_0 = 0$ (surface) and

$I(\tau_0,\mu<0) = 0$ (boundary condition — no inward flux @ surface)

$$I(\tau,\mu<0) = \int_{\tau}^{0} e^{-(t-\tau)/\mu} \frac{S(t)}{\mu}\, dt$$

To do this integral, we need $S(t)$

We know that for $\tau \gg 1$, we expect $S \sim B$

We can Taylor expand about this to account for variation w/ depth (still assume isotropic)

$$S(t) \sim B(\tau) + (t-\tau) \left.\frac{\partial B}{\partial \tau}\right|_{\tau} + \ldots$$

outgoing:

$$I(\tau, \mu > 0) = \frac{1}{\mu} \int_\tau^\infty e^{-(t-\tau)/\mu} \left[B(\tau) + (t-\tau) \frac{\partial B}{\partial \tau} \right] dt$$

$$\xi \equiv \frac{t-\tau}{\mu} \qquad d\xi = \frac{dt}{\mu}$$

$$= \int_0^\infty e^{-\xi} \left[B(\tau) + \mu \xi \frac{\partial B}{\partial \tau} \right] d\xi$$

$$= - B(\tau) e^{-\xi} \Big|_0^\infty + \mu \frac{\partial B}{\partial \tau} \underbrace{\Gamma(2)}$$

$$\int_0^\infty x e^{-x} dx = \Gamma(2) = 1$$

$$= B(\tau) + \mu \frac{\partial B}{\partial \tau}$$

Ingoing:

$$I(\tau, \mu < 0) = \frac{1}{\mu} \int_\tau^0 e^{-(t-\tau)/\mu} \left[B(\tau) + (t-\tau) \frac{\partial B}{\partial \tau} \right] dt$$

$$= \int_0^{-\tau/\mu} e^{-\xi} \left[B(\tau) + \mu \xi \frac{\partial B}{\partial \tau} \right] d\xi$$

$$= - B(\tau) e^{-\xi} \Big|_0^{-\tau/\mu} + \mu \frac{\partial B}{\partial t} \left\{ \frac{e^{-\xi}}{-1} (\xi + 1) \right. \Big|_0^{-\tau/\mu}$$

$$= B(\tau)\left(1 - e^{\tau/\mu}\right) + \mu \frac{\partial B}{\partial \tau} \left[1 - e^{\tau/\mu} \left(-\frac{\tau}{\mu} + 1 \right) \right]$$

now is τ is large (which is what we want — avoid the atmosphere)

$$e^{\tau/\mu} \to 0 \quad (\text{since } \mu < 0)$$

$$\therefore \quad I(\tau, \mu < 0) \sim B(\tau) + \mu \frac{\partial B}{\partial \tau}$$

Note that

$$I(\tau, \mu<0) \sim I(\tau, \mu>0) \; !$$

now $\frac{\partial B}{\partial \tau} > 0$ (T increases inward), so since $I = B(\tau) + \mu \frac{\partial B}{\partial \tau}$

we see I is slightly greater for outward directions ($\mu > 0$) than for inward ($\mu < 0$)

∴ There is a small anisotropy in I resulting from the small T gradient in the star — this can transport energy!

Note: $\frac{dT}{dr} \sim \frac{\Delta T}{\Delta r} \sim \frac{T_c}{R_\odot} = \frac{1.5 \times 10^7 \, K}{7 \times 10^{10} \, cm} = 2 \times 10^{-4} \; K/cm$!

Recall that flux is

$$F = \int_\Omega I \cos\theta \, d\Omega = 2\pi \int_{-1}^{1} I \mu \, d\mu$$

using $I = B(\tau) + \mu \frac{\partial B(\tau)}{\partial \tau}$

$$F = 2\pi \int_{-1}^{1} \Big[\underbrace{B(\tau)}_{\text{even -- cancels out!}} + \mu \frac{\partial B(\tau)}{\partial \tau} \Big] \mu \, d\mu$$

$$= 2\pi \frac{\partial B(\tau)}{\partial \tau} \frac{\mu^3}{3} \Big|_{-1}^{1} = \frac{4\pi}{3} \frac{\partial B(\tau)}{\partial \tau} \qquad \text{(energy/time/area)}$$

B depends only on T, so $\frac{\partial B}{\partial \tau}$ will be $\propto \frac{dT}{d\tau}$

Diffusion (this applies for $\tau \gg 1$ — away from the atmosphere)

taking $F_\nu = \frac{4\pi}{3} \frac{\partial B_\nu}{\partial \tau_\nu}$ (putting frequency dependence back in)

$$\frac{\partial}{\partial \tau_\nu} = -\frac{1}{\kappa_\nu \rho} \frac{\partial}{\partial r}$$

vertical optical depth

$$F_\nu = -\frac{4\pi}{3} \frac{1}{\kappa_\nu \rho} \frac{\partial B_\nu}{\partial r} = -\frac{4\pi}{3} \frac{1}{\kappa_\nu \rho} \underbrace{\frac{\partial B_\nu}{\partial T} \frac{\partial T}{\partial r}}_{\text{chain rule}}$$

$$\therefore \; F_\nu = -\frac{4\pi}{3} \frac{1}{\kappa_\nu \rho} \frac{\partial B_\nu}{\partial T} \frac{dT}{dr}$$

we will define the <u>Rosseland mean opacity</u>

$$\frac{1}{\kappa} \equiv \frac{\int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial B_\nu}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu}$$

this is sometimes referred to as a "gray" opacity, since all the frequency info has been averaged out

then
$$F = -\frac{4\pi}{3} \frac{1}{\rho \kappa} \frac{dT}{dr} \underbrace{\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu}$$

$$= \frac{\partial}{\partial T} \int_0^\infty B_\nu d\nu = \frac{\partial}{\partial T} \frac{\sigma T^4}{\pi}$$

$$= \frac{ac}{\pi} T^3$$

$$\sigma = \frac{ac}{4}$$

Then finally

$$F = -\frac{4ac}{3}\frac{1}{\bar{\kappa}\rho}T^3\frac{dT}{dr}$$

$$L = 4\pi r^2 F(r) = -\frac{(4\pi r^2)\,4ac}{3}\frac{1}{\bar{\kappa}\rho}T^3\frac{dT}{dr}$$ ← luminosity carried by photons

$$= -\frac{16\pi a c r^2}{3\bar{\kappa}\rho}T^3\frac{dT}{dr}$$

this is of the form we wrote earlier:

$$F = -\mathcal{D}\nabla U = -\mathcal{D}\nabla a T^4$$

w/ $\mathcal{D} = \frac{c}{3\bar{\kappa}\rho}$ — recall that $\kappa\rho$ has units of cm^{-1} (inverse of mean free path)

$\therefore$ $\mathcal{D}$ has units of cm^2/s

The Lagrangian form:

$$L(r) = -\frac{(4\pi r^2)^2 a c}{3\bar{\kappa}}\frac{dT^4}{dM}$$

∇'s : funny astronomer notation ...

Note: we can write $\frac{1}{3} a \frac{dT^4}{dM}$ as $\frac{dP_\gamma}{dM}$

from HSE: $$\frac{dP}{dr} = -\frac{GM\rho}{r^2}$$

$$-\frac{r}{P}\frac{dP}{dr} = \frac{GM\rho}{rP} = -\frac{d\log P}{d\log r}$$

dividing by $\frac{d\log T}{d\log r}$, we have

$$\underbrace{\frac{d\log P}{d\log T}}_{\equiv \nabla^{-1}} = -\frac{GM\rho}{rP}\left(\frac{d\log T}{d\log r}\right)^{-1}$$

$$\therefore \quad \nabla = -\frac{r^2 P}{GM\rho}\frac{1}{T}\frac{dT}{dr}$$ $\leftarrow$ this is the actual log slope of T vs. P in the star

nothing radiation-specific here

note ∇ is positive, since $\frac{dT}{dr} < 0$

Our radiation equation

$$L = - \frac{16\pi a c r^2}{3\bar{\kappa}\rho} T^3 \frac{dT}{dr}$$

becomes

$$L_r = - \frac{16\pi a c r^2}{3\bar{\kappa}\rho} T^3 \left[- \frac{GM\rho T}{r^2 P} \nabla \right]$$

$$= \frac{16\pi a c\, G M(r)}{3 P \bar{\kappa}} T^4 \nabla$$

this is the luminosity that can be carried by radiation

For radiation only, we define

$$\nabla_{rad} \equiv \left(\frac{d\log T}{d\log P} \right)_{rad}$$

— this is the log slope of T vs. P if <u>all</u> the luminosity is carried by radiation (the L is L_{total})

if $\nabla = \nabla_{rad}$ then $L = L_{rad}$ and $L_{conv.} = 0$ (no need for convection)

if $\nabla_{rad} > \nabla$ then $L > L_{rad}$ and convection must play a role

needed gradient to carry L

actual gradient

Atmospheres (following HKT § 4.3)

What should our T BC be?

we've been using $T(R_*) = 0$ — not very good

— Doesn't tell us the effective T of the star

Start with our radiation flux:

$$F_\nu = -\frac{4\pi}{3} \frac{1}{\kappa_\nu \rho} \frac{\partial B_\nu}{\partial r} = \frac{L_\nu}{4\pi r^2}$$

recall from the EOS discussion that

$$P_\gamma = \frac{1}{3} \underbrace{\rho e_\gamma}_{\text{we've been calling this } U \text{ here}}$$

so

$$P_{rad} = \frac{4\pi}{3c} B_\nu \quad \left(\text{since } U = \frac{4\pi}{c} \overset{\text{isotropic}}{I} \text{ and } I \sim B \text{ in LTE}\right)$$

$$\therefore \quad \frac{\partial P_{rad,\nu}}{\partial r} = \frac{4\pi}{3c} \frac{\partial B_\nu}{\partial r} = -\frac{\kappa_\nu \rho}{c} \frac{L_\nu}{4\pi r^2}$$

now define an average opacity (not the Rosseland mean)

$$\kappa \equiv \frac{1}{L} \int_0^\infty \kappa_\nu L_\nu \, d\nu$$

then the gray equation is

$$\frac{dP_{rad}}{dr} = -\frac{\kappa \rho L}{4\pi r^2 c} = -\frac{\kappa \rho}{c} F_{rad}$$

20.

We care about the surface, so we take $r \sim R_*$ and we know that

$$L_* = 4\pi R_*^2 \sigma T_{eff}^4$$

$\uparrow$ photosphere radius

the photosphere is defined as the layer we see — it can be ν dependent

Note that T_{eff} may depend on frequency,

How does R_* relate to the surface where $\tau = 0$?

$$P_{rad} = - \int_{\text{surface}}^{\text{some point inside}} \frac{F_{rad}}{c} \kappa\rho\, dr = + \int_{\tau=0}^{\tau} \frac{F_{rad}}{c}\, d\tau$$

since $d\tau \equiv -\kappa\rho\, dr$

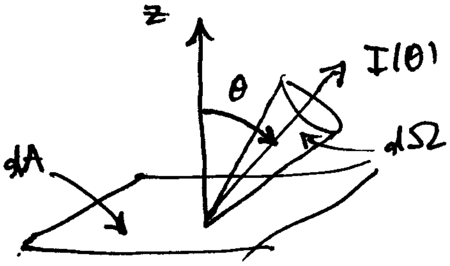
this gives

$$P_{rad}(\tau) = \frac{F_{rad}}{c}\tau + P_{rad}(\tau=0) = \frac{\sigma T_{eff}^4}{c}\tau + P_{rad}(\tau=0)$$

Here we are assuming that $F_{rad} = \frac{L_*}{4\pi R_*^2}$ = const — this is true in the plane parallel approximation when there are no sources in the atmosphere

In this case, we expect $dF_{rad}/d\tau = 0$ — radiative equilibrium.

The pressure is just the momentum flux
(momentum / unit time / unit area)



The momentum flux along the ray $I(\theta)$ (at angle θ) is just dF_ν/c

(since a photon carries momentum $p = E/c$)

$$\therefore \quad P_{rad} = \int_\Omega \frac{dF_\nu}{c} \cos\theta \, d\Omega = \frac{1}{c} \int_\Omega \frac{dE}{dA\,dt} \cos\theta \, d\Omega$$

$$= \frac{1}{c} \int_\Omega I(\theta) \cos^2\theta \, d\Omega$$

↑ momentum flux/ pressure

Notice: we now have 3 quantities that are moments of I

$$U = \frac{2\pi}{c} \int_{-1}^{1} I(\mu)\, d\mu$$

(assuming azimuthal symmetry)

$$F_{rad} = 2\pi \int_{-1}^{1} I(\mu)\, \mu \, d\mu$$

$$P_{rad} = \frac{2\pi}{c} \int_{-1}^{1} I(\mu)\, \mu^2 \, d\mu$$

21a.

Note: we can find the pressure relation

$$\frac{dP_{rad}}{dr} = -\frac{\kappa\rho}{c} F_{rad}$$

starting w/ the radiation transfer equation

$$\mu \frac{dI}{d\tau} = I - S$$

integrating over $d\Omega$

$$2\pi \left\{ \frac{d}{d\tau} \int I \mu \, d\mu = \int I \, d\mu - \int S \, d\mu \right\}$$

take S to be isotropic

$$\frac{dF}{d\tau} = 4\pi \langle I \rangle - 4\pi S$$

This shows $\langle I \rangle = S$ — radiative equilibrium — implies $dF/d\tau = 0$

now multiply by μ and integrate again

$$c \frac{dP}{d\tau} = F \quad \left(\text{since } S \text{ is isotropic, } \int S \mu \, d\mu = 0\right)$$

$$\therefore \frac{dP}{d\tau} = \frac{F}{c}$$

$$\frac{dP}{dr} = -\frac{\kappa\rho}{c} F$$

22. <u>The Eddington approximation</u> (this follows C&O Ch 9)

We need to find the constant (@ $\tau=0$) in our expression

$$P_{rad}(\tau) = \frac{\sigma T_{eff}^4}{c}\tau + P_{rad}(\tau=0)$$

Eddington: decompose I into inward I_{in} and outward I_{out}

impose BC that $I_{in}(\tau=0)=0$

require that $P=\frac{1}{3}U$ still holds everywhere
(true at $\tau>1$ when I becomes nearly isotropic, but near the surface? ...)

using $P_{rad}(\tau=0)=\frac{1}{3}U$

and recalling $\langle I\rangle = \frac{1}{4\pi}\int I\,d\Omega = \frac{1}{2}\int_{-1}^{1} I(\mu)\,d\mu = \frac{c}{4\pi}U$

we have $P_{rad} = \frac{4\pi}{3c}\langle I\rangle$

our moments are

$$\langle I\rangle = \frac{1}{2}\int_{-1}^{1} I(\mu)\,d\mu = \frac{1}{2}\left\{\int_{-1}^{0} \cancel{I_{in}}^{\,0}\,d\mu + \int_{0}^{1} I_{out}\,d\mu\right\} = \frac{I_{out}}{2}$$

$$F = 2\pi\int_{-1}^{1} I(\mu)\,\mu\,d\mu = 2\pi\left\{\int_{-1}^{0} \cancel{I_{in}}^{\,0}\,\mu\,d\mu + \int_{0}^{1} I_{out}\,\mu\,d\mu\right\} = \pi I_{out}$$

$$\therefore\ I_{out} = \frac{F}{\pi}$$

$$\langle I\rangle = \frac{F}{2\pi}$$

$$P_{rad}(\tau=0) = \frac{4\pi}{3c}\left(\frac{F}{2\pi}\right) = \frac{2}{3}\frac{F}{c}$$

Finally, our full expression is

$$P_{rad}(\tau) = \frac{\sigma T_{eff}^4}{c}\tau + \frac{2}{3}\frac{F}{c}$$

$$= \frac{\sigma}{c} T_{eff}^4 \left(\tau + \frac{2}{3}\right) \qquad \left(\text{using } F = \sigma T_{eff}^4\right)$$

in the plane-parallel approximation, the surface flux is the same as the flux at deeper layers

and since $P_{rad} = \frac{1}{3} a T^4$

$$\frac{1}{3} a T^4 = \frac{\sigma}{c} T_{eff}^4 \left(\tau + \frac{2}{3}\right)$$

w/ $a = \frac{4\sigma}{c}$ we get

$$T^4 = \frac{1}{2}\left(1 + \frac{3}{2}\tau\right) T_{eff}^4$$

this shows that the photosphere lies at a depth of $\tau = \frac{2}{3}$ ($T = T_{eff}$ there)

The surface BC should be $T(\tau = 0) = \frac{T_{eff}}{2^{1/4}} \sim .84\, T_{eff}$

Notice also that $\tau = \frac{2}{3} \sim 1$ — this means that the photosphere is approx. 1 mean free path into the Sun

Eddington luminosity

Start w/ HSE:

$$\frac{dP}{dr} = -\rho g$$

Consider just the outer layers (again, plane-parallel), the $g = g_s = \text{const}$

$$-\frac{1}{\rho\kappa}\frac{dP}{dz} = \frac{g_s}{\kappa}$$

$$P(\tau) = g_s \int_0^{\tau} \frac{d\tau}{\kappa}$$

take $\kappa = \text{constant}$ (gray)

$$P(\tau) = \frac{g_s}{\kappa}\tau + P(\tau=0)$$

@ the photosphere, $\tau = \frac{2}{3}$,

$$P_p = \frac{2}{3}\frac{g_s}{\kappa} + P(\tau=0)$$

↑ photosphere

Take $P(\tau=0) = P_\gamma(\tau=0)$ matter how little impact, then

$$P_p = \frac{2}{3}\frac{g_s}{\kappa} + \frac{2}{3c}F(\tau=0) = \frac{2}{3}\frac{g_s}{\kappa} + \frac{2}{3c}\frac{L_*}{4\pi R_*^2}$$

noting $g_s = \frac{GM_*}{R_*^2} \longrightarrow R_*^2 = \frac{GM_*}{g_s}$

$$P_p = \frac{2}{3}\frac{g_s}{\kappa} + \frac{2}{3c}\frac{L_* g_s}{4\pi GM_*} = \frac{2}{3}\frac{g_s}{\kappa}\left(1 + \frac{\kappa L_*}{4\pi c GM_*}\right)$$

This last term is small — except for massive stars

consider P_γ dominating over gravity

$$-\frac{dP_\gamma}{dr} > g_s \rho$$

if radiation transport is in play, then

$$L = -\frac{4\pi R_*^2 c}{\kappa \rho}\frac{dP_\gamma}{dr}$$

and ∴ $$\frac{\kappa \rho L}{4\pi \cancel{R_*^2} c} > \frac{GM_*}{\cancel{R_*^2}}\cancel{\rho}$$

$$L > \underbrace{\frac{4\pi c GM}{\kappa}}_{\text{Eddington luminosity}}$$

if $L > L_{Edd}$, we get mass loss

usually κ here is electron scattering, $\kappa_e \sim 0.34\,cm^2/g$

$$\frac{L_{Edd}}{L_\odot} \sim 3.5\times10^4 \left(\frac{M}{M_\odot}\right)$$

incidentally, this is the biggest L you can find on the MS

if we are in the regime $L \ll L_{Edd}$, then

$$P_p \sim \frac{2g_s}{3\kappa}$$

and we can find the density @ the photosphere