Preliminaries (following HKT Ch 1)

We'll build up the ideas needed to do basic stellar evolution

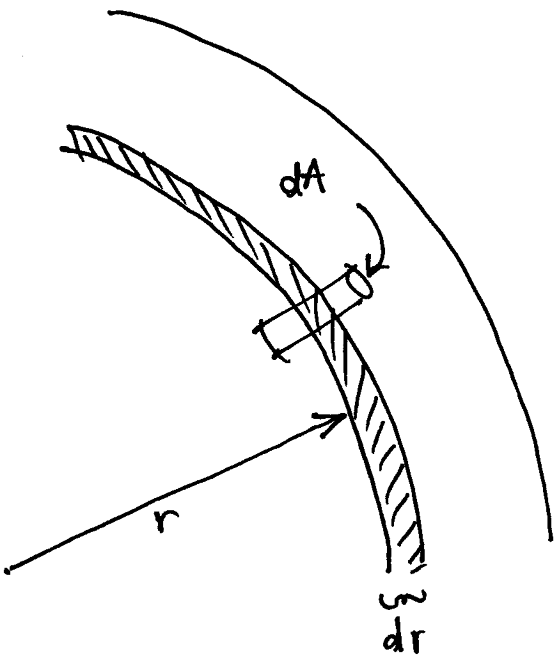
basic thoughts

- stars shine steadily – changes generally not seen in history
- fossil evidence: Sun must be around for billions of years
- stars radiate energy → they must evolve

Stability & HSE

assume: spherical, non-rotating, non-magnetic, isolated star...

if it is stable, then no net accelerations (internal motions average out)



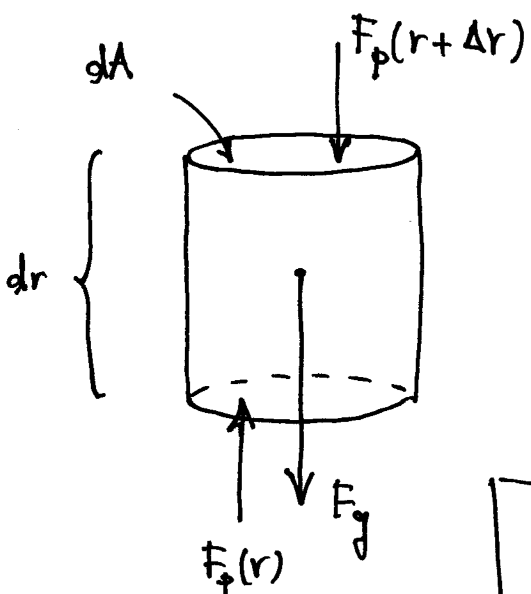
mass inside of r is

$$M(r) = \int_0^r 4\pi r'^2 \rho(r')\, dr'$$

alternately

$$\frac{dM}{dr} = 4\pi r^2 \rho \quad \text{— mass continuity}$$

consider the forces on a small cylindrical element



body force: $F_g = -\rho |g| dr\, dA$

surface forces:

$$F_p(r+\Delta r) = -P(r+dr)\, dA$$

$$F_p(r) = +P(r)\, dA$$

Note: later we'll see why we can neglect accelerations

Newton's law

$$\sum F = \underset{\substack{\uparrow \\ \text{stable}}}{ma} = 0 = -P(r+dr)\,dA + P(r)\,dA - \rho|g|\,dr\,dA$$

$$\therefore \quad \frac{P(r+dr) - P(r)}{dr} = -\rho|g| = \frac{dP}{dr}$$

equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\rho \underset{\text{magnitude}}{g}$$

this is the second equation of stellar structure

Note $|g| > 0$, $\rho \geq 0$, so $\frac{dP}{dr} \leq 0$

pressure decreases outward

$\therefore$ pressure gradient balances gravity

4.

Alternate view – take a global, not local approach

Look at perturbations to total energy – HSE will represent an extremum

Gravitational potential

from Newton, we know $\Omega = -\dfrac{G m_1 m_2}{r}$

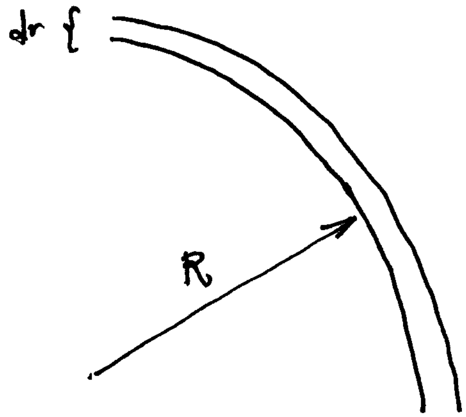
convention is negative

think of $-\Omega$ as the energy required to separate the masses to infinity

$\Omega = 0$ is when masses are infinitely far away

Consider how tightly bound a shell at the surface is

$$d\Omega = -\frac{GM(r)}{r}\,dM$$



for the entire star, we integrate over all shells

$$\Omega = -\int_0^M \frac{GM(r)}{r}\,dM$$

notice we are working in mass coordinates – those tend to be more natural, and give a Lagrangian view

The result of this integral will be something like

$$\Omega = -q\frac{GM^2}{R}$$

where $q \sim O(1)$ depends on the distribution of mass in the star

Ex: constant density

$$dM = 4\pi r^2 \rho \, dr \qquad \rho = \text{constant}$$

$$\Omega = -4\pi \int_0^R \frac{GM(r)}{r} r^2 \rho \, dr$$

$$M(r) = \frac{4}{3}\pi r^3 \rho$$

$$\therefore \Omega = -\frac{(4\pi)^2}{3} G\rho^2 \int_0^R r^4 \, dr$$

$$= -\frac{(4\pi)^2}{3} G\rho^2 \frac{R^5}{5}$$

$$= -\left(\frac{4\pi}{3}\rho R^3\right)^2 \frac{3}{5}\frac{G}{R} = -\underbrace{\frac{3}{5}}_{q=3/5}\frac{GM^2}{R}$$

This can be thought of as a lower limit on q

What about kinetic energy?

we have both

- internal (microscopic)
- macroscopic (turbulence, convection, ...)

consider macroscopic to be negligible, then

$$W = \int_M e\,dM + \Omega = U + \Omega$$

total

specific internal (your text uses E here)

for equilibrium, we want W to be a stationary point

consider adiabatic motions, infinitesimal

no heat transfer between fluid elements

we want a extrema

$$(\delta W)_{ad} = 0 = (\delta U)_{ad} + (\delta \Omega)_{ad}$$

$$U \rightarrow U + \delta U = U + \delta \int_M e\,dM = U + \int_M \delta e\,dM$$

we are following a Lagrangian description

slightly different from text

first law: $dq = 0 = de + P\,d\left(\frac{1}{\rho}\right)$

$$\therefore \delta e = -P\,\delta\left(\frac{1}{\rho}\right) = +\frac{P}{\rho^2}\delta\rho$$

Now $\rho = \dfrac{dM}{4\pi r^2 dr}$ (continuity)

$$= \frac{dM}{d\left(\frac{4}{3}\pi r^3\right)}$$

consider $\rho + \delta\rho$ — assuming spherical symmetry, we have only radial motions

$$\rho + \delta\rho = \frac{dM}{d\left(\frac{4}{3}\pi (r+\delta r)^3\right)} \sim \frac{dM}{\underbrace{d\left(\frac{4}{3}\pi r^3\right) + d\left(4\pi r^2 \delta r\right)}_{\text{binomial}}}$$

$$= \rho \left[1 + \frac{d(4\pi r^2 \delta r)}{d\left(\frac{4}{3}\pi r^3\right)}\right]^{-1} \sim \rho \left(1 - \frac{d(4\pi r^2 \delta r)}{d\left(\frac{4}{3}\pi r^3\right)}\right)$$

$$\therefore \; \delta\rho = -\rho \frac{d(4\pi r^2 \delta r)}{d\left(\frac{4}{3}\pi r^3\right)}$$

and $\delta e = \dfrac{P}{\rho^2}\delta\rho = -\dfrac{P}{\rho}\dfrac{d(4\pi r^2 \delta r)}{d\left(\frac{4}{3}\pi r^3\right)}$

and $(\delta U)_{ad} = -\displaystyle\int_M \frac{P}{\rho} \frac{d(4\pi r^2 \delta r)}{\underbrace{dM/\rho}_{d\left(\frac{4}{3}\pi r^3\right)}}$ since $\rho = \dfrac{dM}{dV}$

$$= -\int P \frac{d(4\pi r^2 \delta r)}{dM} \, dM$$

Boundary conditions:

$$\delta r\,(M(r)=0) = 0 \quad \text{— no motion at center}$$

$$P_s = P(M(r)=M) = 0 \quad \text{— surface (zero BC on } P)$$

integrate by parts

$$\delta U = -\int P \frac{d(4\pi r^2 \delta r)}{dM}\, dM = -\int d\left(\cancelto{0 \text{ by BCs}}{P \frac{4\pi r^2 \delta r}{dM}}\right) dM$$

$$+ \int \frac{dP}{dM}(4\pi r^2 \delta r)\, dM$$

$$\therefore (\delta U)_{ad} = \int \frac{dP}{dM}(4\pi r^2 \delta r)\, dM$$

Now the potential:

$$\Omega \to \Omega + \delta\Omega = -\int_M \frac{GM(r)}{r+\delta r}\, dM$$

$$\sim -\int_M \frac{GM(r)}{r}\, dM + \underbrace{\int \frac{GM(r)}{r^2}\,\delta r\, dM}_{\delta\Omega}$$

$$\therefore (\delta W)_{ad} = \int_M \left[\frac{dP}{dM} 4\pi r^2 + \frac{GM(r)}{r^2}\right] \delta r\, dM = 0$$

the only way for this to be true generally is for

$$\frac{dP}{dM} = -\frac{GM(r)}{4\pi r^4} \quad \Big\} \text{ Lagrangian form of HSE}$$

(HKT guides you to explore $\delta^2 W$ in problem 1.11)

Virial theorem

your text takes a particle-based approach

we'll take an alternate view, starting w/ HSE (= stability)

$$\frac{dP}{dM} = -\frac{GM}{4\pi r^4}$$

multiply by volume $\left(V = \frac{4}{3}\pi r^3\right)$ and integrate

$$\int_0^{P(R)} V\,dP = -\frac{1}{3}\int_0^M \frac{GM\,dM}{r} = \frac{1}{3}\Omega$$

integrate by parts

$$\int_0^{P(r)} V\,dP = VP\Big|_0^{P(r)} - \int_0^{V(r)} P\,dV$$

if we consider the full star, then

$$P(R) = 0 \qquad V(0) = 0$$

and using $dV = \frac{dM}{\rho}$

$$-3\int_0^M \frac{P}{\rho}\,dM = \Omega \qquad \text{Virial theorem}$$

Applications of the Virial theorem

Global energetics

consider a γ-law equation of state

$$P = (\gamma - 1)\rho e \qquad \text{(popular in hydrodynamics)}$$

for an ideal gas, $\gamma = c_p/c_v$ (doesn't need to be monatomic)

for monatomic, $\gamma = 5/3$, $e = \frac{3}{2}\frac{P}{\rho}$ (should look familiar)

for radiation or completely relativistic Fermi gas, $\gamma = 4/3$

Now:
$$3\int_0^M \frac{P}{\rho}\, dM = -\Omega$$

using γ-law

$$3(\gamma-1)\int_0^M e\, dM = 3(\gamma-1)\underbrace{U}_{\text{total internal energy}}$$

$\therefore$ Virial theorem says $3(\gamma-1)U + \Omega = 0$

Total energy is $W = U + \Omega \longrightarrow U = W - \Omega$

$$\therefore \quad 3(\gamma-1)(W-\Omega) + \Omega = 0$$

$$3(\gamma-1)W = +(3\gamma-4)\Omega$$

or
$$W = \frac{3\gamma-4}{3(\gamma-1)}\Omega$$

Stability requires that we are bound : $W < 0$

since $\Omega < 0$, we need $\frac{3\gamma - 4}{3(\gamma - 1)} > 0$ or $\gamma > 4/3$

We will see in some circumstances, we get $\gamma \to 4/3$

(collapse)

Timescales

Kelvin-Helmholtz

- this will be important when considering star formation
- also useful for energy arguments in powering Sun

starting w/ $W = \frac{3\gamma-4}{3(\gamma-1)} \Omega$

if we consider the star to collapse a bit, then W changes

$$\Delta W = \frac{3\gamma-4}{3(\gamma-1)} \Delta\Omega \qquad (\text{we take } \gamma = \text{constant})$$

assume HSE maintained (collapse is slow)

$$\Delta\Omega = \frac{GM^2}{R^2} \Delta R < 0 \qquad (\text{since } \Delta R < 0)$$

energy is liberated!

Where does the energy go? Virial theorem again

$$\Delta U = -\frac{\Delta\Omega}{3(\gamma-1)}$$

for $\gamma = \frac{5}{3}$, $\Delta U = -\frac{1}{2}\Delta\Omega$

$\frac{1}{2}$ is radiated away, $\frac{1}{2}$ goes into heating the star

Look at what this is saying

- star collapses a bit
- U increases — star gets hotter
- W decreases – only $\frac{1}{2}$ of $\Delta\Omega$ went into W, the other $\frac{1}{2}$ was lost!
- $\frac{1}{2}\Delta\Omega$ is radiated away

Loss of energy = hotter star!

This is effectively a negative specific heat

$$c_x \equiv \left.\frac{dq}{dT}\right|_x$$

$$dq = de + p\,d(1/\rho)$$

$$\therefore,\ \text{e.g. } c_v = \left.\frac{dq}{dT}\right|_\rho = \left.\frac{\partial e}{\partial T}\right|_\rho$$

We now want to understand the lifetimes of a star

Gravitational lifetime of the Sun

for $\gamma = \frac{5}{3}$, $\frac{1}{2}$ of the energy from contraction can be radiated

Presently, $\Omega \sim -\frac{GM^2}{R}$

so the amount of energy radiated to date (since the Sun formed and $R \to \infty$) is

$$E_{rad} \sim \frac{1}{2} \frac{GM^2}{R}$$

the present-day luminosity of the Sun is $L = 4 \times 10^{33}$ erg/s

$$\tau_{grav} \sim \frac{E_{rad}}{L} = \frac{1}{2} \frac{GM^2}{RL}$$

this is how long the Sun could shine by radiating this store of energy at a rate L

$$= \frac{1}{2} \frac{6.67 \times 10^{-8}\,\mathrm{dyn\,cm^2\,g^{-2}} \cdot (2 \times 10^{33}\,\mathrm{g})^2}{7 \times 10^{10}\,\mathrm{cm} \cdot 4 \times 10^{33}\,\mathrm{erg/s}}$$

$$= 4.7 \times 10^{14}\,\mathrm{s} = \underline{\underline{1.5 \times 10^{7}\,\mathrm{yr}}}$$

much shorter than fossil record!

This is called the Kelvin-Helmholtz timescale

How much must the Sun contract / year to provide its present luminosity?

$$L = \frac{d\Omega}{dt} = \frac{d\Omega}{dr}\frac{dr}{dt}$$

$$\Omega \sim -\frac{GM^2}{R}$$

$$\frac{d\Omega}{dr} = \frac{GM^2}{R^2}$$

$$\therefore \frac{dr}{dt} = \frac{L}{\frac{d\Omega}{dr}} = \frac{LR^2}{GM^2}$$

$$= \frac{4\times10^{33}\,\mathrm{erg/s}\cdot(7\times10^{10}\,\mathrm{cm})^2}{6.67\times10^{-8}\,\mathrm{dyn\,cm^2\,g^{-2}}\,(2\times10^{33}\,\mathrm{g})^2}$$

$$= 7.2\times10^{-5}\,\mathrm{cm/s}$$

$$= 2200\,\mathrm{cm/yr}$$

this would by very hard to observe

No.

What is the chemical lifetime of the Sun?

Assume Sun is completely H

How many H atoms?

$$N_H = \frac{M_\odot}{m_p} = \frac{2\times10^{33}\,g}{1.67\times10^{-24}\,g} \sim 10^{57}$$

What's a typical energy we can get from chemical reactions?

$\sim 1\,eV$ (maybe up to 10 eV)

$$E_{chemical} = 10^{57}\,eV \cdot 1.6\times10^{-12}\,erg/eV = 1.6\times10^{45}\,erg$$

The chemical lifetime of the Sun is

$$\tau_{chemical} \sim \frac{E_{chemical}}{L_\odot} = \frac{1.6\times10^{45}\,erg}{4\times10^{33}\,erg/s}$$

$$= 0.4\times10^{12}\,s \sim 10^4\,yr\,!$$

this is incredibly short

What is the temperature in the interior of the Sun?

let's see what the Virial theorem says

$$\mathcal{E} = \frac{3}{2} n k T = \frac{3}{2} \frac{\rho k T}{\mu m}$$

$\mathcal{E}$: energy/volume (not specific energy)

n: # density

μ = mean molecular weight / ion (more on this soon)

~ 1 for stellar mix

$$U = \int \mathcal{E} \, dV$$

we are dealing w/ energy / cm^3

$$= \frac{3}{2} \frac{\rho k T}{\mu m} V = \frac{3}{2} \frac{M k T}{\mu m} \qquad (\rho V = M)$$

assuming constant properties

or think as

$$\int \mathcal{E} dV = \int e \, dM$$

since $dM = \rho \, dV$

$$= \int \frac{3}{2} \frac{kT}{\mu m} dM$$

$$= \frac{3}{2} \frac{k \langle T \rangle}{\mu m} M$$

w/ $\langle T \rangle = \dfrac{\int T \, dM}{M}$

Virial theorem: $3(\gamma - 1) U + \Omega = 0$

$\gamma = \frac{5}{3}$ so $U = -\frac{1}{2} \Omega$

assume uniform density,

$$\Omega = -\frac{3}{5} \frac{G M^2}{R}$$

$$\therefore \frac{3}{2} \frac{M k T}{\mu m} = \frac{3}{10} \frac{G M^2}{R} \longrightarrow T = \frac{G M \mu m}{5 k R}$$

so

$$T = \frac{6.67 \times 10^{-8} \, \text{dyn cm}^2 \, \text{g}^{-2} \cdot 2 \times 10^{33} \, \text{g} \cdot 1 \cdot 1.67 \times 10^{-24} \, \text{g}}{5 \cdot 1.38 \times 10^{-16} \, \text{erg/K} \cdot 7 \times 10^{10} \, \text{cm}} = 4.6 \times 10^{6} \, \text{K}$$

or compactly, $T \propto M^{2/3} \rho^{1/3} \mu$

alternately, from HSE

$$\frac{dP}{dM} = -\frac{GM}{4\pi r^4}$$

this says $P \sim \frac{GM^2}{4\pi R^4} \sim \frac{\rho k T}{\mu m} \sim \frac{MkT}{\mu m R^3}$

$$\therefore \; T \sim \frac{G\mu M m}{4\pi k R}$$ — same scaling, w/ different constant

consider a contracting star, so M is fixed, then

$T \sim 1/R$ — star heats up as it contracts

HKT has a figure of T in $\log T - \log \rho$ plane

[Note: this is the "average" temperature — whatever that means]

This T is high enough to ionize atoms:

$$kT = 1.38 \times 10^{-16}\ eV/K \cdot 4 \times 10^6 K = 6 \times 10^{-10}\ erg = 400\ eV$$

we can assume everything is fully ionized in stellar interiors constant density solar model

$\rho =$ constant implies

$$\frac{M_*}{R_*^3} = \frac{M(r)}{r^3}$$

Lagrangian HSE:

$$\frac{dP}{dM} = -\frac{GM(r)}{4\pi r^4} = -\frac{GM_*}{4\pi R_*^4}\left(\frac{M(r)}{M_*}\right)^{-1/3}$$

BC:

$M(r=0) = 0$

$M(r=R_*) = M_*$

$$\int_{P_c}^{P} dP = -\int_0^{M(r)} \frac{GM_*}{4\pi R_*^4}\left(\frac{M(r)}{M_*}\right)^{-1/3} dM$$

$$P - P_c = -\frac{GM_*^2}{4\pi R_*^4}\,\frac{3}{2}\left(\frac{M(r)}{M_*}\right)^{2/3}\Bigg|_0^{M(r)}$$

$$= -\frac{3GM_*^2}{8\pi R^*_4}\left(\frac{M(r)}{M_*}\right)^{2/3}$$

now if $M(r) = M_*$ then $P = 0$ (surface of Sun)

then $P_c = \dfrac{3GM_*^2}{8\pi R_*^4}$ $\longleftarrow$ this is effectively a lower limit on P_c since ρ always decreases w/ r

and $P = P_c\left[1 - \left(\frac{M(r)}{M_*}\right)^{2/3}\right] = P_c\left[1 - \left(\frac{r}{R_*}\right)^2\right]$

evaluating finds $P_c = 10^{15}$ dyn/cm^2 in the Sun

Defining composition

the "Stellar Physics" text by Ed Brown has a very nice discussion on all of this

Consider the ideal gas law:

$$P = nkT$$

density

by definition, n = # of particles / unit volume

Here we need the total # of particles, regardless of what they are

$$n = \frac{\rho}{\bar{m}}$$

average particle mass

notice that $\bar{m} = \mu m_u$

mean molecular weight

Consider completely ionized ^{4}He

what is $\bar{m}$?

insignificant

$$\bar{m} = \frac{1}{3}(4m_u + \cancel{2m_e}) \sim \frac{4}{3} m_u \,, \quad \text{so} \quad \mu = \frac{4}{3}$$

3 particles:
ion + 2 electrons

So the electron mass is negligible, but their # matters a lot!

Alternately

$$P = nkT = \underset{\uparrow \text{ ions}}{n_I} kT + n_e kT$$

charge neutrality, $n_e = Z n_I$ (analogous for multiple species, but they'll be a sum)

$$P = (Z+1) n_I kT$$

what is n_I?

$$\rho = \underbrace{A m_u}_{\text{mass of ion}} n_I + m_e n_e \sim A m_u n_I$$

$$\therefore\ n_I \sim \frac{\rho}{A m_u}$$

and $P = \dfrac{(Z+1)}{A m_u} \rho kT$ w/ $\mu = \dfrac{A}{Z+1}$

$\therefore\ \mu = 4/3$ for ^4He

What about generally?

consider a gas of atoms, ions, and electrons

$$n = n_I + n_e$$

total # density; atoms and ions

$$P = P_I + P_e = \frac{\rho kT}{\mu m_u} = \overbrace{\frac{\rho kT}{\mu_I m_u}}^{n_I kT} + \overbrace{\frac{\rho kT}{\mu_e m_u}}^{n_e kT}$$

$$\therefore \frac{1}{\mu} = \frac{1}{\mu_I} + \frac{1}{\mu_e}$$

for a single species we denote the mass fraction as

$$X_k = \rho_k/\rho$$

which implies $\sum_k X_k = 1$

the # density of species k (ion/atom) is

$$(n_I)_k = \frac{\rho X_k}{A_k m_u}$$

atomic weight of species k

and $n_I = \sum_k (n_I)_k$

$$\therefore \frac{1}{\mu_I} = \sum_k \frac{X_k}{A_k}$$

[note: we implicitly neglect $m_e \ll m_u$]

What about electrons?

identify nuclei k w/ proton # Z_k and mass # A_k

$$\rho_{total} = \sum_k A_k m_u (n_I)_k + \cancel{\sum Z_k y_k m_e (n_e)_k}^{\,0} \qquad \text{since } m_e \ll m_u$$

ionization fraction

but charge neutrality says

$$(n_e)_k = y_k Z_k (n_I)_k$$

ionization fraction $0 \leq y_k \leq 1$

recall that $(n_I)_k = \dfrac{(\rho_I)_k}{A_k m_u} = \dfrac{X_k \rho}{A_k m_u}$

$$\therefore\ n_e = \sum_k (n_e)_k = \sum_k \frac{X_k \rho}{A_k m_u} y_k Z_k$$

$$= \frac{\rho}{m_u} \underbrace{\sum_k y_k \frac{X_k Z_k}{A_k}}_{\equiv 1/\mu_e} = \frac{\rho}{\mu_e m_u}$$

$\equiv 1/\mu_e$ — mean molecular weight per free electron

we'll usually take $y_k = 1$ (completely ionized)

$$n = \frac{\rho}{\mu m_u} = n_I + n_e = \frac{\rho}{\mu_I m_u} + \frac{\rho}{\mu_e m_u}$$

sometimes we talk about electron fraction, $Y_e \equiv \dfrac{1}{\mu_e}$

24.

Notation

X = mass fraction of H

Y = mass fraction of ^{4}He

Z = mass fraction of metals

$$X + Y + Z = 1$$

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assume complete ionization ($y_k = 1$ — good inside stars)

assume $Z \ll 1$

$$\mu_e^{-1} = X + \frac{1}{2} Y \simeq X + \frac{1}{2}(1 - X) = \frac{1+X}{2} \quad (Z \to 0)$$

$\therefore$ $\mu_e \sim \dfrac{2}{1+X}$ inside stars

$$\mu_I^{-1} = X + \frac{Y}{4} \simeq X + \frac{1}{4}(1-X) = \frac{1+3X}{4}$$

$$\mu_I \sim \frac{4}{1+3X}$$

$$\mu^{-1} = \frac{1}{\mu_I} + \frac{1}{\mu_e} = \frac{1+3X}{4} + \frac{1+X}{2} = \frac{3+5X}{4}$$

$$\mu \sim \frac{4}{3+5X}$$

Now we can compute central T in the constant density model

$$P_c = \frac{3GM_*^2}{8\pi R_*^4} = \frac{\rho kT}{\mu m_u} \sim \frac{3M_*}{4\pi R_*^3}\frac{kT}{\mu m_u}$$

$$\therefore T \sim \frac{1}{2}\frac{GM_*}{R_*}\frac{\mu m_u}{k} \sim 1.2\times10^7 \mu \left(\frac{M}{M_\odot}\right)\left(\frac{R}{R_\odot}\right)^{-1} K$$

~0.6 for solar

This is better than the Virial average because we used a real pressure profile

Next up: Energy transport

(following Prialnik)

consider the composition of a star:

stars are made up of gas{H, He, metals, electrons} and photons

radiation will be treated as a "photon gas"

frequent collisions occur between ions, electrons, and photons
- collisions give rise to <u>thermodynamic equilibrium</u>
  - single T characterizes the distribution of particles

ex: a free ideal gas in thermodynamic equilibrium has a Maxwell distribution (more later!...)

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mean free path is important

$$\lambda_\gamma = \underbrace{\frac{1}{n\sigma}}_{\text{this describes the absorbers}} = \frac{1}{\kappa\rho}$$

[note: some texts use κ instead of $\kappa\rho$]

opacity has lots of atomic physics, details of scattering.
In many situations, electron scattering dominates, w/

$$\kappa \sim 1 \text{ cm}^2/\text{g}$$

then for $\rho \sim \langle\rho_\odot\rangle \sim 1\ \text{g/cm}^3$, we have $\lambda_\gamma \sim 1$ cm

so $\lambda_\gamma \ll R_\odot$!

With $\lambda_{mfp} \ll R_*$ the equilibrium can be local and T can vary throughout the star

Further if the time between collisions is $\ll$ macroscopic timescales (luminosity changes, expansion/contraction, ...) then the equilibrium can change in time

Equilibrium between matter and radiation also occurs
— radiation achieves Planck spectrum (blackbody)

Local thermodynamic equilibrium, LTE

[LTE means that we can use a single T at each point in the star to describe both matter and radiation]

Note: LTE is not always achieved, e.g. sunlight w/ 6000 K spectrum passing through our atmosphere

Together this means we can describe the thermodynamic structure of a star by $\underbrace{\rho, T, X_k}_{2+N \text{ quantities}}$

We've eliminated gravitational and chemical energy as the "fuel" for powering the sun

We'll study fusion in detail in Ch. 6

Some (obvious things)

- Higher temp and ρ make it easier to fuse
- Heavier nuclei need even higher T — you need to overcome the Coulomb barrier
- There is a limit to how much energy we can get out — ^{56}Ni is most tightly bound nucleus, beyond that we need to <u>input</u> energy to fuse

Thermal balance: energy released by reactions is carried away

$\epsilon \equiv$ power generated / gram [erg/g/s]
(energy generation rate)

in spherical shell, <u>power</u> generated is

$$4\pi r^2 \rho \epsilon \, dr = \epsilon \, dM$$

In particular, we will use the fact that the nuclear timescale is long compared to the dynamical timescale / thermal timescale to assume that the star achieves equilibrium much faster than burning

$$\underbrace{\tau_{KH}}_{\text{already estimated}} \ll \tau_{nuc}$$ (follow Prialnik §2.8)

$$\tau_{nuc} \sim \frac{\epsilon M c^2}{L}$$

ϵ is the fraction of a nuclei rest energy that is released in other forms of energy

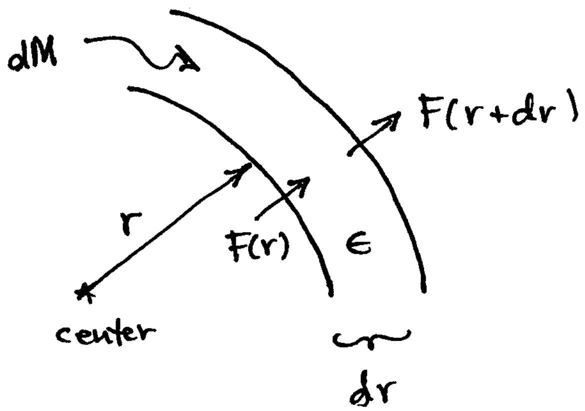
We get ~ few MeV per nucleon (at most) or $\epsilon \sim 5 \times 10^{-3}$ ($m_{nucleon} \sim 1$ GeV)

$$\tau_{nuc} \sim \frac{5\times10^{-3} \cdot 2\times10^{33}\,g \cdot (3\times10^{10}\,cm/s)^2}{4\times10^{33}\,erg/s}$$

$$\sim 5\times10^{18}\,s \sim 5\times10^{10}\,yr$$ (longer than the age of the Universe)

The energy is transported, so we have a flux $F(r)$

$F(r)$ = flux leaving shell ($erg/cm^2/s$)



convention: $F > 0$ means radially outward

define luminosity through shell $L(r) = 4\pi r^2 F(r)$

$$L(r+dr) - L(r) = \underbrace{4\pi r^2 \rho \, dr}_{dM} \, \epsilon$$

(some sources use q instead of ϵ)

$\epsilon \equiv$ energy / mass / time

this gives: $\dfrac{dL}{dr} = 4\pi r^2 \rho \epsilon$

the Lagrangian form is $\dfrac{dL}{dM} = \epsilon$

boundary conditions: $L(r=0) = 0$

The form of ϵ depends on the reactions, but we can parameterize as

$$\epsilon = \epsilon_0 \rho^{\lambda} T^{\nu}$$

$$\lambda = \begin{cases} 1 & \text{for 2-body reactions} \\ 2 & \text{for 3-body reactions} \end{cases}$$

$$\nu \sim \begin{cases} 4 & \text{for pp burning} \\ 15 & \text{for CNO} \\ 40 & \text{for } 3\text{-}\alpha \end{cases}$$

for H burning, we have

$$4\,{}^1H \rightarrow {}^4He + \text{stuff}$$

$$\Delta E = (3.97\, m_p - 4\, m_p)\, c^2$$

$$= 4.5 \times 10^{-5}\ \text{erg}$$

per nucleon

$$\epsilon_0 = \frac{\Delta E}{4 m_p} = 6 \times 10^{18}\ \text{erg/g} \quad (\text{taking into account } \nu \text{ losses})$$

So far we have

$$\frac{dr}{dM} = \frac{1}{4\pi r^2 \rho}$$ mass conservation

$$\frac{dP}{dM} = -\frac{GM(r)}{4\pi r^4}$$ dynamic equilibrium (basically momentum conservation)

$$\frac{dL}{dM} = \epsilon$$ energy conservation

this system is not closed

those 2 would have time derivatives, but the timescales are small compared to main-sequence

We need to express how the energy is transported

We can have

- conduction (not important in normal stars)
- radiation
- convection

Radiation

photon energy density (we'll derive this later via integrating the Planck function)

$$u = aT^4$$

Fick's law of diffusion: great approximation when we are optically thick)

$$F(r) = -\underbrace{D}_{\text{diffusion coefficient} \sim \frac{1}{\kappa}} \frac{d(aT^4)}{dr}$$

κ ← opacity

later we'll see

$$D = \frac{c}{3\kappa\rho}$$

Important opacities:

$\kappa \sim 0.2(1+X)\ \mathrm{cm^2/g}$ – electron scattering (important in high mass stars)

$\kappa \sim Z(1+X)\rho T^{-7/2}$ – Kramers opacity (involves atoms) – important in low-mass stars

$$\frac{dr}{dM} = \frac{1}{4\pi r^2 \rho}$$

$$\frac{dP}{dM} = -\frac{GM(r)}{4\pi r^4}$$

$$\frac{dL}{dM} = \epsilon$$

$$F(r) = -D\,\frac{d(aT^4)}{dr}$$

we need microphysics:

$$\left.\begin{matrix}\kappa \\ P \\ \epsilon\end{matrix}\right\} \text{ all as functions of } \rho, T, X_k$$

then we can solve for r, P, L, and T as functions of M