Applications of polytropes

Eddington standard model (HKT §7.2.7)

- builds on the polytrope solution by incorporating an energy equation

We'll assume radiation transport dominates.

What is ∇ for a mix of gas and radiation pressure?

$$\nabla \equiv \frac{d\log T}{d\log P} = \frac{3}{16\pi ac}\frac{P\bar{\kappa}}{T^4}\frac{L}{GM}$$

consider gas pressure:

$$P_\gamma = \frac{1}{3}aT^4 \longrightarrow T^4 = \frac{3P_\gamma}{a} \rightarrow dT = \frac{3}{4aT^3}\,dP_\gamma$$

so we can write

$$\nabla = \frac{P}{T}\frac{dT}{dP} = \frac{P}{T}\frac{dT}{dP_\gamma}\frac{dP_\gamma}{dP} = \frac{P}{T}\frac{3}{4aT^3}\frac{dP_\gamma}{dP} = \frac{1}{4}\frac{P}{P_\gamma}\frac{dP_\gamma}{dP}$$

$$\therefore \frac{dP_\gamma}{dP} = \frac{1}{4\pi c}\frac{\bar{\kappa}}{G}\frac{L}{M} = \frac{L_\star\bar{\kappa}}{4\pi c\, GM_\star}\frac{L/L_\star}{M/M_\star}$$

Now consider our energy equation

$$\frac{dL}{dM} = \epsilon$$

we can define an average energy generation rate as

$$\langle \epsilon(r) \rangle = \frac{\int_0^r \epsilon \, dM}{\int_0^r dM} = \frac{L(r)}{M(r)}$$

and then

$$\langle \epsilon(R_\star) \rangle = \frac{L_\star}{M_\star}$$

define $\eta(r) = \dfrac{\langle \epsilon(r) \rangle}{\langle \epsilon(R_\star) \rangle} = \dfrac{L/L_\star}{M/M_\star}$

then we have

$$\frac{dP_\gamma}{dP} = \frac{L_\star}{4\pi c G M_\star} \bar{\kappa}(r) \eta(r)$$

if we take $P(R_\star) = 0$, we have

$$P_\gamma = \int_0^{P(r)} \frac{L_\star}{4\pi c G M_\star} \bar{\kappa}(r) \eta(r) \, dP$$ (integrating from surface inward)

no assumptions so far aside from

- thermal equilibrium
- radiation transport dominates

Now define

$$\langle \bar{\kappa}(r)\eta(r)\rangle = \frac{1}{P(r)} \int_{\underset{P(r=R_*)}{0}}^{P(r)} \bar{\kappa}(r)\eta(r)\, dP$$

then

$$P_\gamma = \frac{L_*}{4\pi c G M_*} \langle \bar{\kappa}(r)\eta(r)\rangle P(r)$$

Now define $\beta \equiv \frac{P_{gas}}{P} \rightarrow 1-\beta = \frac{P_\gamma}{P}$

then

$$1-\beta = \frac{L_*}{4\pi c G M_*} \langle \bar{\kappa}(r)\eta(r)\rangle$$

Now we need to do something about $\langle \bar{\kappa}\eta\rangle$

- This is what Eddington approximated in 1926

To a good approximation,

$$\kappa = \underbrace{\kappa_e}_{\text{electron scattering}} + \underbrace{\kappa_0 \rho T^{-3.5}}_{\text{ion processes}} \quad \rightarrow \quad \kappa(r) \text{ increases w/ } r$$

also

η will be strongly peaked toward the center and fall off from there.

Eddington: take $\kappa\eta$ = constant

Immediate implication: β is constant in the star

Now we can find the T profile

$$P_\gamma = (1-\beta) P_{tot} = \frac{(1-\beta)}{\beta} P_{gas} = \frac{1-\beta}{\beta} \frac{k}{\mu m_0} \rho T = \frac{1}{3} a T^4$$

$$\therefore \; T = \left(\frac{3k}{a\mu m_0} \frac{1-\beta}{\beta} \right)^{1/3} \rho^{1/3}$$

and

$$P = \frac{k}{\mu m_0} \frac{\rho T}{\beta} = \left[\left(\frac{k}{\mu m_0} \right)^4 \frac{3}{a} \frac{1-\beta}{\beta^4} \right]^{1/3} \rho^{4/3}$$

this is an $n=3$ polytrope

if we take composition to be uniform ($\mu =$ const) then
$K =$ const

From polytropes, we have

$$K = \left[\frac{4\pi}{\xi^{n+1} (-\theta')^{n-1}} \right]^{1/n}_{\xi=\xi_1} \frac{G}{n+1} M_*^{1-1/n} R_*^{-1+3/n} \quad \text{(HKT 7.40)}$$

for $n=3$

$$K = \frac{(4\pi)^{1/3}}{4} \frac{G M_*^{2/3}}{\left[\xi^4 (-\theta')^2 \right]_{\xi=\xi_1}}$$

equating:

$$\frac{1-\beta}{\beta^4} = 2.996 \times 10^{-3} \mu^4 \left(\frac{M}{M_\odot} \right)^2$$ — β and M_* are not independent!

We can also find

$$T = 4.62\times10^{6}\, \beta\mu \left(\frac{M}{M_\odot}\right)^{2/3} \rho^{1/3}$$

trends:

- more massive stars are hotter
- more massive stars → more influence P_r has (lower β)

Note: Eddington standard model does not provide numerical values for T, ρ, ...

If we know both M & R, we can get physical values

Also note that this predicts $T \propto \rho^{1/3}$ for radiative stars

6. White dwarf structure

Again from polytropes:

$$K = \left(\frac{4\pi}{\xi^{n+1}(-\theta')^{n+1}} \right)^{1/n}_{\xi=\xi_1} \frac{G}{n+1} M_*^{1-1/n} R_*^{-1+3/n}$$

for a non-relativistic degenerate gas, we know $n = 3/2$

and we found K earlier:

$$P = 10^{13} \left(\frac{\rho / 1\,g/cm^3}{\mu_e} \right)^{5/3} dyn\,cm^{-2}$$

equating the K and using $n = 3/2$, we have

$$\frac{M}{M_\odot} = 2.08 \times 10^{-6} \left(\frac{2}{\mu_e}\right)^5 \left(\frac{R}{R_\odot}\right)^{-3}$$

— this is the WD M-R relation

Note: for relativistic case, $n=3$, the radius cancels out

What about mass in the relativistic case?

$$M = -\left(\frac{1}{4\pi}\right)^{1/2}\left(\frac{n+1}{G}\right)^{3/2} K^{3/2} \rho_c^{(3-n)/2n} \xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}$$

note that the central density dependence goes away for $n=3$

Then using the relativistic EOS:

$$P = 1.2\times10^{15}\left(\frac{\rho/1\,\mathrm{g\,cm^{-3}}}{\mu_e}\right)^{4/3} \mathrm{dyn\ cm^2}$$

we find

$$\frac{M}{M_\odot} = 1.45\left(\frac{2}{\mu_e}\right)^2$$

This is the Chandrasekhar mass

(HKT 7.3.2)

Radiative envelope

e.g. red supergiant : dense core, extended envelope

we'll consider the case where envelope mass is negligible

$$M(r > R_{core}) \sim M_*$$

and all the energy generation is in the core

$$L(r > R_{core}) \sim L_*$$

next we'll assume that convection doesn't exist, then

$$\nabla = \nabla_{rad} = \frac{3}{16\pi a c G} \frac{P\kappa}{T^4} \frac{L_*}{M_*}$$

now we'll assume an ideal gas and an opacity of the form

$$\kappa = \kappa_0 \rho^{v} T^{-s} \quad w/ \quad \rho = \frac{\mu m_u}{k} \frac{P}{T}$$

$$= \kappa_g P^{v} T^{-v-s} \quad w/ \quad \kappa_g = \kappa_0 \left(\frac{\mu m_u}{k}\right)^{v}$$

then we have

$$\nabla = \frac{P}{T}\frac{dT}{dP} = \frac{3}{16\pi a c G} \frac{P}{T^4} \kappa_g P^{v} T^{-v-s} \frac{L_*}{M_*}$$

or $$P^{v} dP = \frac{16\pi a c G}{3\kappa_g} \frac{M_*}{L_*} T^{3+v+s} dT$$

Take a reference T_0 and P_0 (e.g. at the photosphere) and $P(r) > P_0$; $T(r) > T_0$, then

$$\int_P^{P_0} P^{\upsilon}\, dP = \frac{16\pi a c G M_*}{3 \kappa_g L_*} \int_T^{T_0} T^{3+\upsilon+s}\, dT$$

$$\frac{1}{\upsilon+1}\left[P_0^{\upsilon+1} - P^{\upsilon+1}\right] = \frac{16\pi a c G M_*}{3\kappa_g L_*} \frac{1}{4+\upsilon+s}\left[T_0^{4+\upsilon+s} - T^{4+\upsilon+s}\right]$$

(as long as $4+\upsilon+s \neq 0$)

if $\upsilon+s+4 > 0$ and $\upsilon+1 > 0$, then $T(r) \gg T_0$, $P(r) \gg P_0$

since

$$P^{\upsilon+1}\left[1-\left(\frac{P_0}{P}\right)^{\upsilon+1}\right] = \frac{\upsilon+1}{4+\upsilon+s} \frac{16\pi a c G M_*}{3\kappa_g L_*} T^{4+\upsilon+s}\left[1-\left(\frac{T_0}{T}\right)^{4+\upsilon+s}\right]$$

and we can take

$$P^{\upsilon+1} \sim \frac{\upsilon+1}{4+\upsilon+s} \frac{16\pi a c G M_*}{3\kappa_g L_*} T^{4+\upsilon+s}$$

Notice that $P(T)$ is independent of the photosphere values in the interior

(consistent w/ the idea that we could use $T=0$ at surface)

This works for Kramers' opacity ($\upsilon=1$, $s=3.5$)
and electron scattering ($\upsilon=s=0$)

Note: for H^- opacity (important in low mass stars)

$\upsilon=\frac{1}{2}$, $s=-9$

and the interior does connect to the surface T_0, P_0

Back to assuming $\nu+s+4>0$, $\nu+1>0$

$$\nabla(r)=\frac{d\log T}{d\log P}\longrightarrow\frac{\nu+1}{\nu+s+4}\equiv\frac{1}{1+n_{eff}}$$

remember also how $\nabla_{ad}=1-\frac{1}{\Gamma_2}$

(here $n_{eff}=\frac{s+3}{\nu+1}$ is effective polytropic index)

This is because we can write

$P=K'\,T^{1+n_{eff}}$ from our $P^{\nu+1}\sim T^{4+\nu+s}$ relation

w/ $$K'=\left[\frac{1}{1+n_{eff}}\,\frac{16\pi ac\,GM_\star}{3\kappa_0 L_\star}\right]^{1/\nu+1}\left(\frac{k}{\mu m_u}\right)^{\nu/\nu+1}$$ (replaced κ_g w/ κ_0)

recall for a polytrope

$$P=K\rho^{1+1/n}$$

ideal gas: $T=\frac{\mu m_u}{k}\frac{P}{\rho}$

$$\therefore\ P=K'\left(\frac{\mu m_u}{k}\right)^{n+1}\frac{P^{n+1}}{\rho^{n+1}}\longrightarrow P=(K')^{-n}\left(\frac{k}{\mu m_u}\right)^{n/n+1}\rho^{1+1/n}$$

this relates K' to the polytrope K

Note: this is only the envelope, so some other way of connecting to the rest of the star through a different polytrope is needed

Consider a completely convective star (§7.3.3)

- we will still have a thin radiative envelope where radiation escapes through the photosphere
- We can use the previous model to estimate the depth of the radiative layer and connect to an underlying convective star

Cool stars have H^- opacity

$$\kappa_{H^-} \sim 2.5\times10^{-31} \left(\frac{Z}{0.02}\right) \rho^{1/2}\, T^{9} \quad \mathrm{cm}^2/\mathrm{g}$$

as we saw, this combination of exponents means the interior is sensitive to what happens @ surface.

photosphere conditions: $T_p = T_{eff}$

$$P_p \approx \frac{2 g_s}{3\kappa_p} \quad \text{(as found in our gray atm)}$$

now start w/

$$(*) \qquad P^{\nu+1} = \frac{\nu+1}{4+\nu+s}\,\frac{\alpha}{\kappa_g}\,T^{4+\nu+s}\,\frac{\left[1-\left(\frac{T_0}{T}\right)^{4+\nu+s}\right]}{\left[1-\left(\frac{P_0}{P}\right)^{\nu+1}\right]}$$

$$\alpha \equiv \frac{16\pi a c\, G M_*}{3 L_*}$$

What is ∇?

$$P^{\nu+1} - P_0^{\nu+1} = \frac{\nu+1}{4+\nu+s} \frac{\alpha}{\kappa_g} \left[T^{4+\nu+s} - T_0^{4+\nu+s} \right]$$

$$\cancel{(\nu+1)} P^{\nu} dP = \frac{\cancel{\nu+1}}{\cancel{4+\nu+s}} \frac{\alpha}{\kappa_g} \cancel{(4+\nu+s)} T^{3+\nu+s} dT$$

$$\frac{dT}{dP} = \frac{\kappa_g}{\alpha} \frac{P^{\nu}}{T^{3+\nu+s}}$$

(we can't use the $\frac{1}{1+n_{eff}}$ here because we can't neglect P_0, T_0)

then

$$\nabla = \frac{P}{T}\frac{dT}{dP} = \frac{\kappa_g P^{\nu+1}}{\alpha T^{4+\nu+s}}$$

we can define a photosphere ∇:

$$\nabla_p = \frac{\kappa_g}{\alpha} \frac{P_p^{\nu+1}}{T_p^{4+\nu+s}}$$

(this is HKT Eq 7.128)

then starting w/ ∇ and using the general expression (*)

$$\nabla = \frac{\kappa_g}{\alpha} \frac{P^{\nu+1}}{T^{4+\nu+s}} = \frac{\cancel{\kappa_g}}{\cancel{\alpha}} \frac{1}{1+n_{eff}} \frac{\cancel{\alpha}}{\cancel{\kappa_g}} \frac{\left[1 - (T_{eff}/T)^{4+\nu+s}\right]}{\left[1 - (P_p/P)^{\nu+1}\right]}$$

or

$$\left[1 - \left(\frac{P_p}{P}\right)^{\nu+1}\right] \nabla = \frac{1}{1+n_{eff}} \left[1 - \left(\frac{T_{eff}}{T}\right)^{4+\nu+s}\right]$$

what is P_p/P ?

consider $$\frac{\nabla_p}{\nabla} = \left(\frac{P_p}{P}\right)^{\upsilon+1}\left(\frac{T}{T_{eff}}\right)^{4+\upsilon+s}$$

then
$$\left[1 - \frac{\nabla_p}{\nabla}\left(\frac{T_{eff}}{T}\right)^{4+\upsilon+s}\right]\nabla = \frac{1}{1+n_{eff}}\left[1 - \left(\frac{T_{eff}}{T}\right)^{4+\upsilon+s}\right]$$

finally
$$\nabla = \frac{1}{1+n_{eff}} + \left(\frac{T_{eff}}{T}\right)^{4+\upsilon+s}\left[\nabla_p - \frac{1}{1+n_{eff}}\right]$$ (this is HKT 7.127)

Now $\kappa_g = \kappa_0 \left(\frac{\mu m_0}{k}\right)^{\upsilon}$

opacity coeff w/ P, T

opacity coeff w/ ρ, T

$$\nabla_p = \frac{3\kappa_g L_*}{16\pi a c G M_*}\frac{P_p^{\upsilon+1}}{T_{eff}^{4+\upsilon+s}} = \frac{3\kappa_0 L_*}{16\pi a c G M_*}\left(\frac{\mu m_0}{k}\right)^{\upsilon}\frac{P_p^{\upsilon+1}}{T_{eff}^{4+\upsilon+s}}$$

photosphere opacity: $\kappa_p = \kappa_0 \rho_p^{\upsilon} T_{eff}^{-s}$

$$P_p = \frac{\rho_p k T_{eff}}{\mu m_0} \rightarrow P_p^{\upsilon} = \rho_p^{\upsilon} T_{eff}^{\upsilon}\left(\frac{k}{\mu m_0}\right)^{\upsilon}$$

insert above $P_p^{\upsilon+1} = P_p^{\upsilon} P_p$

$$\therefore \nabla_p = \frac{3L_*}{16\pi a c G M_*}\kappa_0 \rho_p^{\upsilon} \cancel{T_{eff}^{\upsilon}} \frac{P_p}{T_{eff}^{4+\cancel{\upsilon}+s}}$$

$$= \frac{3L_*}{16\pi a c G M_*}\underbrace{\kappa_0 \rho_p^{\upsilon} T_{eff}^{-s}}_{\kappa_p} T_{eff}^{-4} P_p$$

Now taking

$$P_p = \frac{2g_s}{3\kappa_p}, \quad g_s = \frac{GM_*}{R_*^2}, \quad L_* = 4\pi R_*^2 \sigma T_{eff}^4$$

then

$$\nabla_p = \frac{3}{16\pi ac\, GM_*} \kappa_p \cancel{T_{eff}^{-4}} \underbrace{\left[4\pi R_*^2 \sigma \cancel{T_{eff}^4}\right]}_{L_*} P_p \qquad \sigma = \frac{ac}{4}$$

$$= \frac{3}{16} \underbrace{\frac{R_*^2}{GM_*}}_{g_s} \kappa_p \left(\underbrace{\frac{2g_s}{3\kappa_p}}_{P_p}\right)$$

$$= \frac{1}{8} \; !$$

Then for H^- opacity, $n_{eff} = -4$ and $v = \frac{1}{2}, s = 9$

$$\nabla = -\frac{1}{3} + \left(\frac{T_{eff}}{T}\right)^{-4.5} \left[\underbrace{\frac{1}{8}}_{\nabla_p} + \underbrace{\frac{1}{3}}_{-\frac{1}{1+n_{eff}}}\right]$$

$$= -\frac{1}{3} + \frac{11}{24}\left(\frac{T_{eff}}{T}\right)^{-4.5} \longleftarrow \text{HKT Eq 7.129}$$

Now ∇ increases w/ depth, since T increases w/ depth, so at some point we will have $\nabla > \nabla_{ad} \longrightarrow$ convective !

For an ideal gas, $\nabla_{ad} = \frac{2}{5}$ and $P = \rho^{\gamma} \rightarrow n = 3/2$ polytrope

current picture:

- radiative photosphere (which is shallow)
- convective core

For fully convective, we must have

$$K' = \left(\frac{k}{\mu m_0}\right)^{n+1} K^{-n} \qquad \text{(ideal gas)}$$

$$P = K' T^{n+1}$$

and we know

$$K = \left[\frac{4\pi}{\xi^{n+1}(-\theta_n')^{n-1}}\right]^{1/n}_{\xi_1} \frac{G}{n+1} M_*^{1-1/n} R_*^{-1+3/n}$$

and $P = K' T^{5/2}$

solving for K':

$$K' = \left(\frac{k}{\mu m_0}\right)^{5/2} \frac{\xi^{5/2}(-\theta')^{1/2}}{4\pi}\Bigg|_{\xi_1} \left(\frac{5}{2}\right)^{3/2} \frac{1}{G^{3/2} M_*^{1/2} R_*^{3/2}}$$

we define

$$E_0 = \left[-\left(\frac{5}{2}\right)^3 \xi^5 \theta'\right]^{1/2}_{\xi_1 (\text{for } n = 3/2)} \sim \underset{\text{using } \xi_1 \text{ and } \theta'|_{\xi_1} \text{ tabulated by Chandra}}{\underline{45.48}}$$

We want relations between M, T_{eff}, and L for fully convective stars.

First: What is the depth of the radiative layer ($\nabla = \nabla_{ad}$)?

$$\nabla = -\frac{1}{3} + \frac{11}{24}\left[\frac{T_{eff}}{T_f}\right]^{-4.5} = \frac{2}{5}$$

T_f is base of atm

∇_{ad} for $\gamma = \frac{5}{3}$ ideal gas

this gives

$$\frac{T_f}{T_{eff}} = \left(\frac{15}{24}\right)^{-2/9} = 1.11$$

— T_f is just 11% higher than T_{eff}, so this is just below the photosphere

similarly P_f can be found in terms of P_p

previously we had

$$\left[1 - \left(\frac{P_p}{P}\right)^{v+1}\right]\nabla = \frac{1}{1+n_{eff}}\left[1 - \left(\frac{T_{eff}}{T}\right)^{4+v+s}\right]$$

and $\frac{\nabla_p}{\nabla} = \left(\frac{P_p}{P}\right)^{v+1}\left(\frac{T}{T_{eff}}\right)^{4+v+s}$

so $$\left(\frac{P}{P_p}\right)^{v+1} = 1 + \frac{1}{1+n_{eff}}\frac{1}{\nabla_p}\left[\left(\frac{T}{T_{eff}}\right)^{4+v+s} - 1\right]$$

using our T_f/T_{eff}, we have $\frac{P_f}{P_p} = 2^{2/3}$

17.

evaluating $K' \sim 3.5\times10^{-4} \dfrac{E_0}{\mu^{5/2}} \left(\dfrac{M_*}{M_\odot}\right)^{-1/2} \left(\dfrac{R_*}{R_\odot}\right)^{-3/2}$

so $K' = K'(M_*, R_*, \mu)$

We can eliminate R_* in favor of $R_* = \left(\dfrac{L_*}{4\pi\sigma T_{eff}^4}\right)^{1/2}$

now $P_p = \dfrac{2g_s}{3\kappa_p} = \dfrac{2}{3}\left(\dfrac{GM_*}{R_*^2}\right)\dfrac{1}{\kappa_0} \rho_p^{-\nu} T_{eff}^{s}$

writing $\rho_p = \dfrac{P_p \mu m_u}{k T_{eff}}$

$$P_p = \frac{2}{3}\left(\frac{GM_*}{R_*^2}\right)\frac{1}{\kappa_0}\left(\frac{k}{\mu m_u}\right)^{\nu} T_{eff}^{s+\nu} P_p^{-\nu}$$

$$\therefore\ P_p = \left[\frac{2}{3}\left(\frac{GM_*}{R_*^2}\right)\frac{1}{\kappa_0}\right]^{1/(\nu+1)} \left(\frac{k}{\mu m_u}\right)^{\nu/(\nu+1)} T_{eff}^{(s+\nu)/(\nu+1)}$$

some more algebra, w/

$P_f = K' T_f^{5/2}$

$P_f = 2^{2/3} P_p \qquad T_f = 1.11 T_{eff}$

and using our K' and P_p expressions, w/ R_* eliminated in favor of T_{eff} and L_* gives

$$T_{eff} \sim 2600\,\mu^{13/51}\left(\frac{M_*}{M_\odot}\right)^{7/51}\left(\frac{L_*}{L_\odot}\right)^{1/102} K$$

18. The exponents here are strange

2600K is a little low, it should be more like 4000 K,

but this will show up on the H-R diagram as essentially a vertical line (for a given mass)

T_{eff} is basically independent of L_*

So completely convective stars follow a nearly ~~so~~ vertical line on the HR diagram.

This applies to proto-stars

The effective temperature cannot fall below this value

These paths are called the Hayashi tracks