Stellar Models

Equations of Stellar Structure

- Stellar structure—no time dependence (for now…)
  - Hydrostatic + thermal equilibrium
  - Spherical/no-rotation
  - No magnetic fields
- Necessary inputs:
  - Stellar mass
  - Composition as a function of r or M(r)
  - Microphysics (all functions of ρ, T, X_k):
    - Equation of state
    - Energy generation rate (i.e. nuclear burning stages)
    - opacity

- We are here:

$$\frac{dP}{dM} = -\frac{GM}{4\pi r^4}$$
$$\frac{dr}{dM} = \frac{1}{4\pi r^2 \rho}$$
$$\frac{dL}{dM} = \epsilon$$

- Choose between radiation and convection:

$$\nabla = \begin{cases} \nabla_{\rm rad} & \text{if } \nabla_{\rm rad} < \nabla_{\rm ad} \\ \nabla_{\rm ad} & \text{otherwise} \end{cases}$$

$$\nabla_{\rm rad} = \frac{3}{16\pi ac} \frac{P\bar{\kappa}}{T^4} \frac{L}{GM}$$

Uniqueness

- Vogt-Russel "theorem"
  - Given the mass and composition, the structure of the star (T, R, L, …) follows (sort of)
  - Turns out that it cannot be proven that unique solutions exist
  - But usually only one of the solutions corresponds to a configuration found in nature

Boundary Conditions

- Four equations = four boundary conditions
- Center BCs:

$$r(M=0)=0$$
$$L(M=0)=0$$

- Surface BCs:

$$P(M=M_\star)=0$$
$$T(M=M_\star)=0$$

- Radiative Zero BC:
  - Ideally we would use some atmospheric model to tell us what the temperature BC is at the surface
  - T change over the whole star is so large, the difference between 0 and the real T_{eff} at surface is smalll
  - To get effective T at the end, use:

$$T_{\text{eff}}=\left(\frac{L}{4\pi R^2\sigma}\right)^{1/4}$$

Polytropes

- *Polytropes provide a simplified stellar model that can be used to tell some approximate behavior of stellar interiors*
- We want to express the relation between pressure and density as:

$$P = K\rho^{1+1/n}$$

  - n is called the polytropic index
- Note: it does not necessarily have to be that the EOS is in this form, rather, the stratification of the star could obey this type of scaling
  - In this sense, the energy equations are implicitly satisfied by giving us that prescribed stratification (e.g. the T being adiabatic)

– Some examples:

- Fully convective (adiabatic):

$$P \propto \rho^{\Gamma_1}$$

- White dwarfs (completely degenerate):

$$P \propto \rho^{5/3} \, \text{or} \, \rho^{4/3}$$

- Pressure is a mix of gas + radiation, but the ratio is constant throughout (this will lead to the Eddington Standard Model later)

Polytropes

- Consider only HSE

$$\frac{dP}{dr} = -\frac{GM}{r^2}\rho$$
$$\frac{r^2}{\rho}\frac{dP}{dr} = -GM$$

- Differentiating again and using continuity:

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dM}{dr} = -4\pi G r^2 \rho$$
$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

- Note that this is similar to Poisson's equation in spherical coords

$$g = -d\phi/dr$$
$$\nabla^2\phi = 4\pi G\rho$$

Polytropes

- Now we make this dimensionless
  - Central density: ρ_c
  - Define: θ such that

$$\rho(r) = \rho_c \theta^n(r)$$

  - Then:

$$P(r) = K\rho_c^{1+1/n}\theta^{n+1}(r) = P_c\theta^{n+1}(r)$$

$$P_c \equiv K\rho_c^{1+1/n}$$

  - Finally, introduce a length scale:

$$r_n^2 = \frac{(n+1)P_c}{4\pi G\rho_c^2}$$

$$r = r_n \xi$$

- Result: Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

- Solutions to this are called “polytropes of index n”

Polytropes

- Consider an ideal gas

$$P = \frac{\rho k T}{\mu m_u} \rightarrow \rho = \frac{\mu m_u P}{kT}$$

- in our polytrope relation:

$$P = K\rho^{1+1/n} \qquad \rho = \rho_c \theta^n$$

- and some algebra…

$$P = K\left(\frac{\mu m_u P}{kT}\right)^{1+1/n}$$

$$1 = K^n \left(\frac{\mu m_u}{k}\right)^{n+1} P \left(\frac{1}{T}\right)^{n+1}$$

$$P = K^{-n}\left(\frac{k}{\mu m_u}\right)^{n+1} T^{n+1} = K' T^{n+1}$$

- We see that θ plays the role of T in this case if μ is constant
  - This will be useful when we look at fully convective stars

Polytropes

- Finally, we need BCs
  - Keeping ρ_c as the central density:

$$\theta(\xi = 0) = 1$$

  - Symmetry in spherical coordinates:

$$d\theta/d\xi|_{\xi=0} = 0$$

- What about the surface
  - Integrating outward, the surface will be defined as the first zero of θ:

$$\xi_1 \, : \, \theta(\xi_1) = 0$$

- The physical radius of the star is then just:

$$R_\star = r_n \xi_1$$

- Solutions that do not diverge (b/c of the symmetry BC) are called E-solutions

Analytic Solutions

- Only n = 0, 1, and 5 have analytic solutions
- n = 0
  - Constant density sphere (compare to what we calculated in class)

$$\rho(r) = \rho_c$$

$$\theta(\xi) = 1 - \frac{\xi^2}{6} \rightarrow \xi_1 = \sqrt{6}$$

$$P(\xi) = P_c\theta(\xi) = P_c\left[1 - (\xi/\xi_1)^2\right]$$

- n = 1

$$\theta(\xi) = \frac{\sin\xi}{\xi} \rightarrow \xi_1 = \pi$$

$$\rho = \rho_c\theta$$

$$P = P_c\theta^2$$

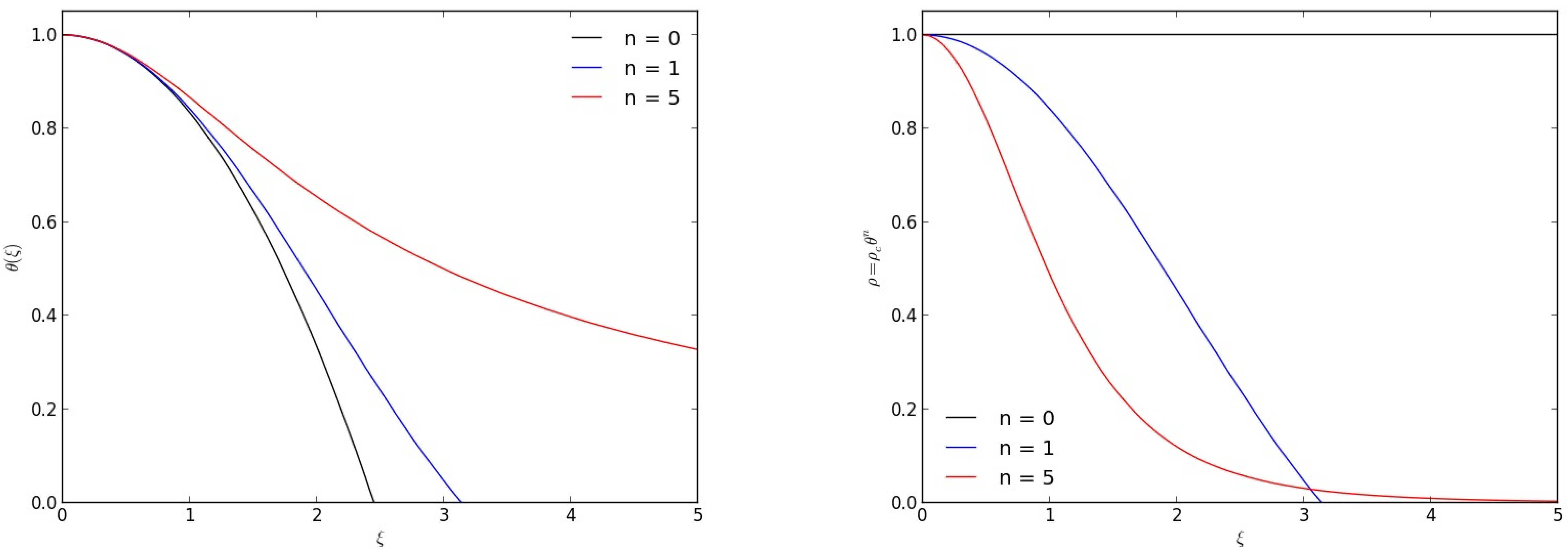
Analytic Solutions

- n = 5
  - Finite mass + central density, but infinite radius

$$\theta(\xi) = \frac{1}{(1+\xi^2/3)^{1/2}} \rightarrow \xi_1 \rightarrow \infty$$

  - Any solutions with n > 5 have infinite mass—not interesting

Analytic Solutions



Making Sense of Solutions

- Working with solutions:
  - Given n and K, we get $\rho(\xi)$ and $P(\xi)$
  - To get $R_\star$ (and therefore physical units) we need K and ρ_c or P_c

$$r_n^2 = \frac{(n+1)P_c}{4\pi G \rho_c^2}$$

$$R_\star = r_n \xi_1$$

- We typically don't have a feel for ρ_c or P_c, but we do know what mass we are interested in
  - We use the mass to tell us what central density we have
  - Then we can evaluate the stellar radius
  - All remaining stellar properties can now be determined (in physical units)

Making Sense of Solutions

- What is M?

$$M(r) = 4\pi \int_0^r {r'}^2 \rho(r') dr'$$

- Dimensionless:

$$M(\xi) = 4\pi r_n^3 \int_0^\xi {\xi'}^2 \rho(\xi') d\xi'$$

$$= 4\pi r_n^3 \int_0^\xi {\xi'}^2 \left(\rho_c \theta^n\right) d\xi'$$

- Substituting in the Lane-Emden eq:

$$M(\xi) = -4\pi r_n^3 \rho_c \int_0^\xi \frac{d}{d\xi'} \left({\xi'}^2 \frac{d\theta}{d\xi'} \right) d\xi'$$

$$= -4\pi r_n^3 \rho_c \, {\xi'}^2 \left(\frac{d\theta}{d\xi'} \right) \Bigg|_{\xi'=\xi}$$

Making Sense of Solutions

- Total mass:

$$M_\star = M(\xi_1)$$

- Substituting in for r_n:

$$M_\star = -\frac{1}{\sqrt{4\pi}}\left(\frac{n+1}{G}\right)^{3/2}\frac{P_c^{3/2}}{\rho_c^2}\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}$$

- Eliminate central pressure in favor of K (compare to Clayton Eq. 2-306)

$$M_\star = -\frac{1}{\sqrt{4\pi}}\left(\frac{n+1}{G}\right)^{3/2}K^{3/2}\rho_c^{(3-n)/2n}\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}$$

- This gives us everything we need to complete the solution
  - Note that for n = 3, M is independent of ρ_c

Making Sense of Solutions

- Our radius relation becomes:

$$R_\star = r_n \xi_1 = \left[\frac{(n+1)}{4\pi G}\right]^{1/2} K^{1/2} \rho_c^{(1-n)/2n} \xi_1$$

- From mass and radius we can get the average density:

$$\bar{\rho} = \frac{M_\star}{\frac{4}{3}\pi R_\star^3} = \frac{3}{\xi_1}\left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_1} \rho_c$$

  - This shows that the average / central density is a function of the polytropic index only (no K).
  - This ratio is a measure of how concentrated the mass of the model is toward the center

Making Sense of Solutions

Table 2-5 Constants of the Lane-Emden functions†

n	ξ_1	$-\xi_1^2\left(\frac{d\phi}{d\xi}\right)_{\xi=\xi_1}$	$\frac{\rho_c}{\bar{\rho}}$
0	2.4494	4.8988	1.0000
0.5	2.7528	3.7871	1.8361
1.0	3.14159	3.14159	3.28987
1.5	3.65375	2.71406	5.99071
2.0	4.35287	2.41105	11.40254
2.5	5.35528	2.18720	23.40646
3.0	6.89685	2.01824	54.1825
3.25	8.01894	1.94980	88.153
3.5	9.53581	1.89056	152.884
4.0	14.97155	1.79723	622.408
4.5	31.83646	1.73780	6,189.47
4.9	169.47	1.7355	934,800
5.0	∞	1.73205	∞

† S. Chandrasekhar, "An Introduction to the Study of Stellar Structure," p. 96; reprinted from the Dover Publications edition, Copyright 1939 by The University of Chicago, as reprinted by permission of The University of Chicago.

Making Sense of Solutions

- Similar expressions can be found for central pressure, T (if ideal gas), K given M and R, …
- General idea: approximate a real star by a polytrope and then we have an approximate measure of its structure
- Interesting values:
  - Completely degenerate, non-relativistic electron gas: n = 3/2
  - Fully relativistic degenerate electron gas: n = 3
  - Fully convective, ideal gas, n = 3/2
  - Star in radiative equilibrium (Eddington standard model), n = 3
- Sadly, analytic solutions do not exist for these interesting cases

Integrating Lane-Emden

- We can integrate our L-E equations using Runge Kutta

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

- Write it as a system of 2 equations
  - New variables:

$$y = \theta$$

$$z = \frac{d\theta}{d\xi}$$

- Now a system of 2 first order ODEs:

$$\frac{dy}{d\xi} = z$$

$$\frac{dz}{d\xi} = -y^n - \frac{2}{\xi}z$$

Integrating Lane-Emden

- Our boundary conditions are

$$\theta(0) = 1 \rightarrow y(0) = 1$$
$$\theta'(0) = 0 \rightarrow z(0) = 0$$

- Note that at ξ = 0 the RHS is undefined
  - Do an expansion about the origin
  - Symmetry: the odd powers go away

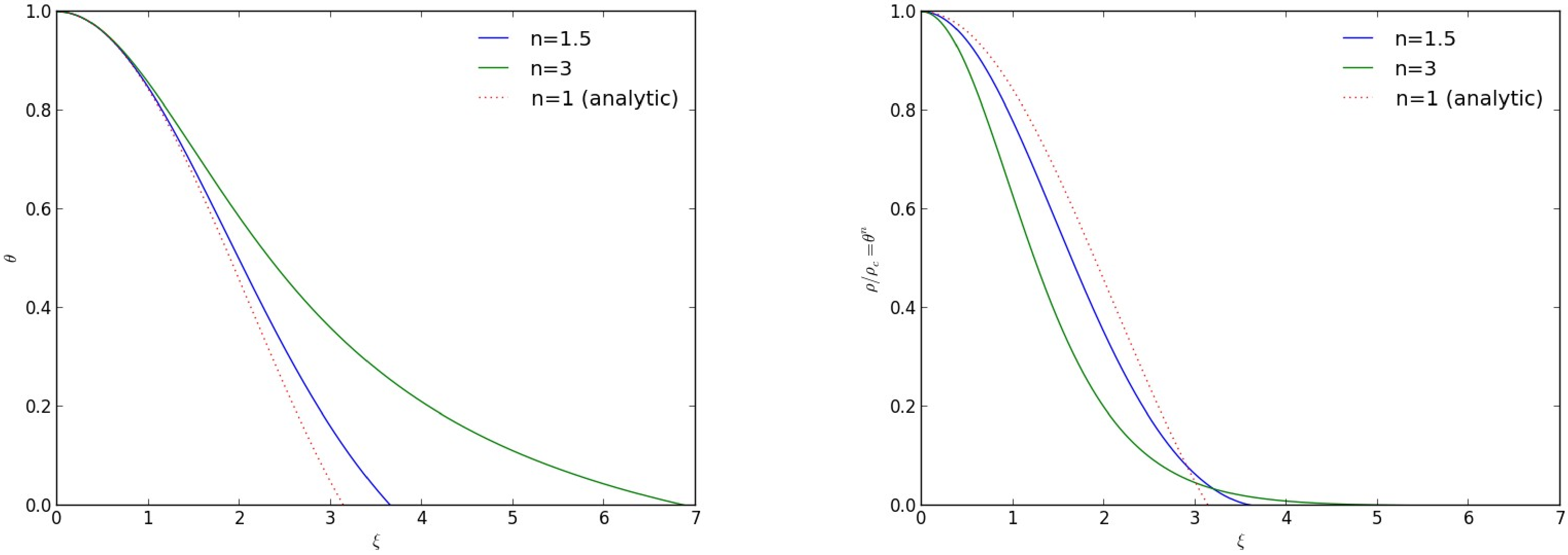
$$\theta(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 + \ldots$$

- The expansion is then

$$\frac{dz}{d\xi} \sim -\frac{1}{3} + \frac{n}{10}\xi^2$$

- When do we stop?
  - We can estimate the point at which our function goes negative and adjust our stepsize to hit it (within some tolerance)

Numerical Solutions



Shooting / Fitting

- Shooting can work with systems of ODEs
- Commonly used with the full equations of stellar structure
  - Central p and T unknown, surface L and R unknown.
  - There we integrate out from the center and in from the surface simultaneously
  - Meet in the middle
  - Adjust parameters to get a match at the middle
  - Iterate

Fitting

- We want to simultaneously integrate our L-E equation in and out
- 2 systems of 2 equations → 4 BCs total
  - Center: $y(\xi = 0) = 1\,, z(\xi = 0) = 0$
  - Surface: $y(\xi = \xi_s) = 0\,, z(\xi = \xi_s) = \alpha$
- Additionally, there are two unknowns:

$$\xi_s\,, \alpha$$

- We are free to choose a fitting point: ξ_f

- Procedure:
  - Integrate from the center outward to the fit point:

$$y_{\rm out}(\xi_f), z_{\rm out}(\xi_f)$$

  - Integrate from the surface (guess) inward to the fit point:

$$y_{\rm in}(\xi_f), z_{\rm in}(\xi_f)$$

  - We want to zero two functions:

$$Y(\alpha, \xi_s) \equiv y_{\rm in}(\xi_f) - y_{\rm out}(\xi_f) = 0$$
$$Z(\alpha, \xi_s) \equiv z_{\rm in}(\xi_f) - z_{\rm out}(\xi_f) = 0$$

Fitting

- Solving this system:
  - Newton method (Taylor expansion):

$$Y(\alpha + \Delta\alpha, \xi_s + \Delta\xi_s) = Y(\alpha, \xi_s) + \frac{\partial Y}{\partial \alpha}\Delta\alpha + \frac{\partial Y}{\partial \xi_s}\Delta\xi_s \sim 0$$

$$Z(\alpha + \Delta\alpha, \xi_s + \Delta\xi_s) = Z(\alpha, \xi_s) + \frac{\partial Z}{\partial \alpha}\Delta\alpha + \frac{\partial Z}{\partial \xi_s}\Delta\xi_s \sim 0$$

  - You need the derivatives
    - These are with the other quantity held constant
    - You need to integrate the system 3 times total
      - 1. integrate with α, ξ_s: Y, Z
      - 2. integrate with $\alpha+\delta\alpha$, ξ_s: Y^α, Z^α
      - 3. integrate with α, $\xi_s+\delta\xi_s$: Y^ξ, Z^ξ
    - Numerical differences:

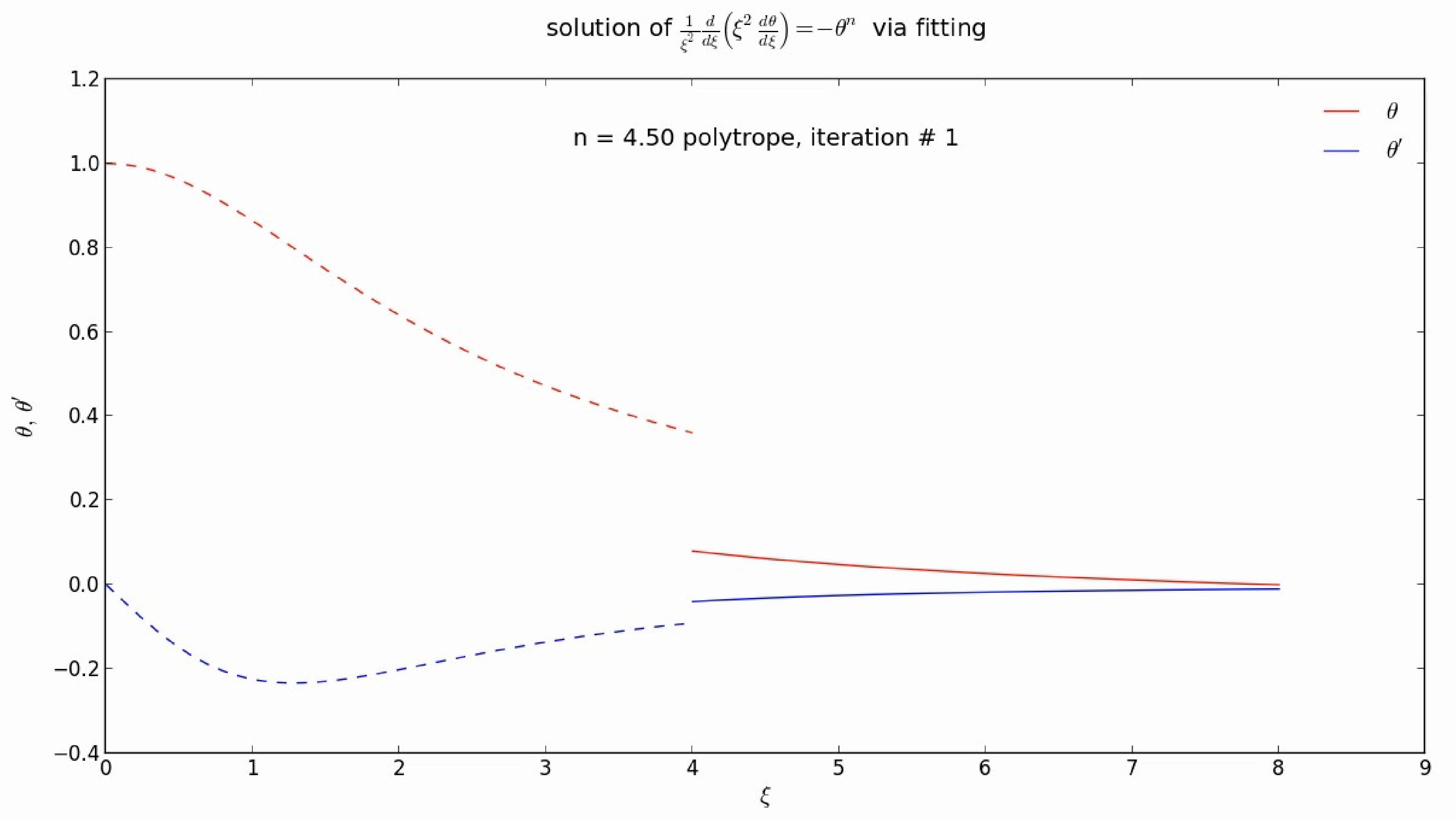
$$\partial Y/\partial\alpha = (Y^\alpha - Y)/\delta\alpha,\ \partial Y/\partial\xi_s = (Y^\xi - Y)/\delta\xi_s, \ldots$$

Fitting

- You should pick the δ's to be small (~10^{-8} relative)
- You can solve for the corrections algebraically
  - $\alpha \rightarrow \alpha + \Delta\alpha\,, \xi_s \rightarrow \xi_s + \Delta\xi_s$
- Note: a really good guess is needed of else you can diverge
- This is really tricky, since the accuracy with which you solve the system comes into play in a non-linear fashion

- Another popular method is the Henyey method—you can explore this for your project…

Fitting



solution of $\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)=-\theta^n$ via fitting
n = 4.50 polytrope, iteration # 1
θ
θ'
θ, θ'
ξ
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
0 1 2 3 4 5 6 7 8 9

Polytrope Summary

- HSE + mass conservation gave us the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

  - Solutions called *polytropes of index n*
  - Assumed equation of state of form:

$$P = K\rho^{1+1/n}$$

  - Expressed density as:

$$\rho = \rho_c\theta^n$$

  - Analytic solutions only for n = 0, 1, and 5
- From solution and with choice of M, we can get:
  - central density, radius, central pressure, …

Polytrope Summary

- Main results:

$$M_\star = -\frac{1}{\sqrt{4\pi}}\left(\frac{n+1}{G}\right)^{3/2} K^{3/2}\rho_c^{(3-n)/2n}\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}$$

$$R_\star = r_0\xi_1 = \left[\frac{(n+1)}{4\pi G}\right]^{1/2} K^{1/2}\rho_c^{(1-n)/2n}\xi_1$$

$$\bar{\rho} = \frac{M_\star}{\frac{4}{3}\pi R_\star^3} = \frac{3}{\xi_1}\left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}\rho_c$$

Eddington Standard Model

- Assume that stars are completely radiative

$$\nabla = \frac{d\ln T}{d\ln P} = \frac{3}{16\pi ac}\frac{P\bar{\kappa}}{T^4}\frac{L}{GM}$$

- Introduce radiation pressure:

$$\nabla = \frac{P}{T}\frac{dT}{dP} = \frac{P}{T}\frac{dT/dr}{dP/dr}$$

$$T^4 = \frac{3}{a}P_\gamma \rightarrow dT = \frac{3}{4aT^3}dP_\gamma$$

$$\therefore \nabla = \frac{P}{T}\frac{3}{4aT^3}\frac{dP_\gamma}{dP} = \frac{1}{4}\frac{P}{P_\gamma}\frac{dP_\gamma}{dP}$$

- We can rewrite the radiation equation as:

$$\frac{dP_\gamma}{dP} = \frac{1}{4\pi c}\frac{\bar{\kappa}}{G}\frac{L}{M} = \frac{L_\star\bar{\kappa}}{4\pi cGM_\star}\frac{L/L_\star}{M/M_\star}$$

Eddington Standard Model

- Now consider the energy generation equation

$$\frac{dL}{dM} = \epsilon$$

  - We define the average energy generation rate as:

$$\langle \epsilon(r) \rangle = \frac{\int_0^r \epsilon dM}{\int_0^r dM} = \frac{L}{M}$$

$$\langle \epsilon(R_\star) \rangle = \frac{L_\star}{M_\star}$$

- Defining the normalized average:

$$\eta(r) = \frac{\langle \epsilon(r) \rangle}{\langle \epsilon(R_\star) \rangle} = \frac{L/L_\star}{M/M_\star}$$

- We have:

$$\frac{dP_\gamma}{dP} = \frac{L_\star}{4\pi c G M_\star} \bar{\kappa}(r)\eta(r)$$

  - Up to now, the only assumption we've made is that we are completely radiative (and in equilibrium)

Eddington Standard Model

- Consider the average of energy rate × opacity over the star

$$\langle \kappa \bar{(r)} \eta \rangle = \frac{1}{P(r)} \int_0^{P(r)} \bar{\kappa} \eta dP$$

- Let's integrate our radiation equation:

$$dP_\gamma = \frac{L}{4\pi c G M_\star} \kappa \eta dP$$

- Integrate from surface (P=0) inward:

$$\int_{P(R_\star)=0}^{P(r)} dP_\gamma = \frac{L}{4\pi c G M_\star} \int_{P(R_\star)=0}^{P(r)} \kappa \eta dP$$

$$P_\gamma(r) = \frac{L_\star}{4\pi c G M_\star} \int_{P(R_\star)=0}^{P(r)} \kappa \eta dP$$

  - Define a new average:

$$\langle \kappa(r) \eta(r) \rangle \equiv \frac{\int_{P(R_\star)=0}^{P(r)} \kappa \eta dP}{\int_{P(R_\star)=0}^{P(r)} dP}$$

$$= \frac{1}{P(r)} \int_0^{P(r)} \kappa \eta dP$$

Eddington Standard Model

- Our integrated equation becomes:

$$P_\gamma(r) = \frac{L_\star}{4\pi c G M_\star} \langle \bar{\kappa}(r)\eta(r) \rangle P(r)$$

- Now introduce: $\beta = P_{\rm gas}/P$

$$1 - \beta(r) = \frac{L_\star}{4\pi c G M_\star} \langle \bar{\kappa}(r)\eta(r) \rangle$$

- We need to do something about transport and energy coefficients

- Opacity
  - $\kappa = \kappa_{es} + \kappa_0 \rho T^{-3.5}$
  - This will increase with r
- Energy generation rate
  - Regardless of the burning, for H, we expect it to be strongly peaked toward the center
- Eddington (1926):
  - Take κη ~ constant (!)
  - Furthermore, it we take the mean molecular weight as constant (good for ZAMS) then β = constant!

Eddington Standard Model

- Let's look at the EOS

$$P_\gamma = (1-\beta)P_{\rm tot} = \frac{1-\beta}{\beta}P_{\rm gas} = \frac{1-\beta}{\beta}\frac{k}{\mu m_u}\rho T = \frac{1}{3}aT^4$$

$$T = \left(\frac{3k}{\mu a m_u}\frac{1-\beta}{\beta}\right)^{1/3}\rho^{1/3}$$

- And…

$$P = \frac{k}{\mu m_u}\frac{\rho T}{\beta} = \left[\left(\frac{k}{\mu m_u}\right)^4 \frac{3}{a}\frac{1-\beta}{\beta^4}\right]^{1/3}\rho^{4/3}$$

  – This is an n=3 polytrope!

Eddington Standard Model

- In this form, we have a constant in our EOS:

$$K = \left[\left(\frac{k}{\mu m_u} \right)^4 \frac{3}{a} \frac{1-\beta}{\beta^4} \right]^{1/3}$$

- If we know n and M, then from the polytrope solutions, we can also get K (HKT 7.40)
  - For n = 3:

$$K = \frac{(4\pi)^{1/3}}{4} \frac{G M_\star^{2/3}}{[\xi^4(-\theta')^2]_{\xi=\xi_1}}$$

- Equating:

$$\frac{1-\beta}{\beta} = 2.996 \times 10^{-3} \mu^4 \left(\frac{M}{M_\odot} \right)^2$$

$$T = 4.62 \times 10^6 \beta \mu \left(\frac{M}{M_\odot} \right)^{2/3} \rho^{1/3}$$

Eddington Standard Model

- Trends:
  - Massive stars have more radiation pressure dominance
  - Massive stars have higher T
- Note: this is best for a ZAMS star—structure changes as the star evolves

Table 7.2. Eddington Standard Model

$\mu^2 M/M_\odot$	β
1.0	0.9970
2.0	0.9885
5.0	0.9412
10.0	0.8463
50.0	0.5066

- The standard model fits a detailed solar ZAMS model well in the interior
- There are departures near the surface, not unexpected though, since this is where the model really is convective, and an n = 3/2 polytrope.

White Dwarfs

- From our solutions of polytropes, we already used the relation of K to M and R:

$$K = \left(\frac{4\pi}{[\xi^{n+1}(-\theta')^{n-1}]_{\xi=\xi_1}} \right)^{1/n} \frac{G}{n+1} M_\star^{1-1/n} R_\star^{-1+3/n}$$

- For a non-relativistic degenerate gas, n = 3/2, and we know K is the quantity we evaluated in our homework:

$$P = 10^{13} \left(\frac{\rho/1\text{g cm}^{-3}}{\mu_e} \right)^{5/3} \text{dyn cm}^{-2}$$

- Equating and taking n = 3/2

$$\frac{M}{M_\odot} = 2.08 \times 10^{-6} \left(\frac{2}{\mu_e} \right)^5 \left(\frac{R}{R_\odot} \right)^{-3}$$

  - This is the WD mass-radius relation we saw previously

White Dwarfs

- For the relativistic case, n = 3

$$K = \left(\frac{4\pi}{[\xi^{n+1}(-\theta')^{n-1}]_{\xi=\xi_1}}\right)^{1/n} \frac{G}{n+1} M_\star^{1-1/n} R_\star^{-1+3/n}$$

  - Radius cancels out
  - Also last time we saw:

$$M_\star = -\frac{1}{\sqrt{4\pi}} \left(\frac{n+1}{G}\right)^{3/2} K^{3/2} \rho_c^{(3-n)/2n} \xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}$$

  no central density dependence here

- Using the relativistic degenerate EOS:

$$P = 1.2 \times 10^{15} \left(\frac{\rho/1\text{g cm}^{-3}}{\mu_e}\right)^{4/3} \text{dyn cm}^{-2}$$

White Dwarfs

- Equating the Ks:

$$\frac{M}{M_\odot} = 1.45 \left(\frac{2}{\mu_e}\right)^2$$

  - This is the Chandrasekhar mass

- A few comments:
  - Real white dwarfs will have a transition between the non-relativistic and extreme relativistic regimes
  - The Eddington model was also n = 3, but note that the EOS K was not constant—it had a mass dependence (through β)

Stellar Envelopes

- For the outer part of a star, we can take L and M to be constant and we can show that under certain circumstances, the envelope can act like a polytrope
  - This gets messy... I'll scan some notes that show the details

- Basic idea:
  - assume convection is negligible:

$$\nabla = \nabla_\mathrm{rad}$$

$$\nabla = \frac{d\log T}{d\log P} = \frac{3}{16\pi acG}\frac{P\kappa}{T^4}\frac{L_\star}{M_\star}$$

  - assume we are an ideal gas, with an opacity:

$$\kappa = \kappa_0 \rho^\nu T^{-s} = \kappa_g P^\nu T^{-\nu-s}$$

  - then we have:

$$P^\nu dP = \frac{16\pi acG}{3\kappa_g}\frac{M_\star}{L_\star}T^{3+\nu+s}dT$$

Stellar Envelopes

- Take a photosphere reference, with $P(r) > P_0$, $T(r) > T_0$
  - integrate from some depth to the photosphere:

$$P^{\nu+1}\left[1-\left(\frac{P_0}{P}\right)^{\nu+1}\right] = \frac{\nu+1}{4+\nu+s}\frac{16\pi acGM_\star}{3\kappa_g L_\star}T^{4+\nu+s}\left[1-\left(\frac{T_0}{T}\right)^{4+\nu+s}\right]$$

- Now notice—if the exponnets are positive, then we are not sensitive to the photosphere conditions:

$$P^{\nu+1} \sim \frac{\nu+1}{4+\nu+s}\frac{16\pi acGM_\star}{3\kappa_g L_\star}T^{4+\nu+s} \qquad \nu+1>0; 4+\nu+s>0$$

  - an important exception to this is H- opacity, which is important in low mass stars

Stellar Envelopes

- From

$$P^{\nu+1} \sim \frac{\nu+1}{4+\nu+s} \frac{16\pi acGM_\star}{3\kappa_g L_\star} T^{4+\nu+s}$$

we have:

$$\nabla(r) \rightarrow \frac{\nu+1}{4+\nu+s} \equiv \frac{1}{1+n_{\mathrm{eff}}}$$

  - thus our radiative envelopes behave as a polytrope

- Our pressure-temperature relation is:

$$P = K' T^{1+n_{\mathrm{eff}}}$$

$$K' = \left(\frac{1}{1+n_{\mathrm{eff}}} \frac{16\pi acGM_\star}{3\kappa_0 L_\star} \right)^{\frac{1}{\nu+1}} \left(\frac{k_B}{\mu m_u} \right)^{\frac{\nu}{\nu+1}}$$

  - This is related to the polytrope K

- This allows us to solve for the structure of the radiative envelope. But we would still need to connect it to a model for the core.

Envelopes

- Fully convective stars will still have a radiative envelope (thin) where the radiation escapes through the photosphere
- We can use the previous model to estimate the depth of the radiative layer, and connect to an underlying convective model

- H^- opacity is:

$$\kappa_{H^-} \sim 2.5 \times 10^{-31} \left(\frac{Z}{0.02}\right) \rho^{1/2} T^9 \ \mathrm{cm}^2 \ \mathrm{g}^{-1}$$

  – This combination of exponents mean that the interior is sensitive to the photosphere
  – Lot's of algebra gives:

$$\nabla = \frac{1}{1+n_{\rm eff}} + \left(\frac{T_{\rm eff}}{T}\right)^{4+\nu+s} \left[\nabla_p - \frac{1}{1+n_{\rm eff}}\right]$$

$$\nabla_p = \frac{3\kappa_g L_\star}{16\pi a c G M_\star} \frac{P_p^{\nu+1}}{T_{\rm eff}^{4+\nu+s}}$$

Envelopes

- Plugging in photospheric values (going back to our gray atmosphere), you can show

$$\nabla_p = \frac{1}{8}$$

$$\nabla = -\frac{1}{3} + \frac{11}{24}\left[\frac{T_{\rm eff}}{T}\right]^{-4.5}$$

  - at some depth, we will find that $\nabla > \nabla_{\rm ad}$ and we are convective

- With a lot of algebra (see your text), we can connect an n = 3/2 polytrope (for the convective interior) to the radiative phosphere, and find a relation between effective temperature, mass, and luminosity

$$T_{\rm eff} \approx 2600\mu^{13/51}\left(\frac{M}{M_\star}\right)^{7/51}\left(\frac{L}{L_\star}\right)^{1/102}\ \mathrm{K}$$

  - note those exponents—this is vertical on the HR diagram
  - effective temperature of fully convective stars is independent of how the energy is generated

Where Are We?

- Last time we worked on polytropes
- Today we saw some applications
- We'll talk next about going beyond the stellar structure equations

- Coming soon:
  - More on instabilities
  - Stellar evolution w/ MESA
  - Low mass vs. high mass evolution

Dynamic Problems

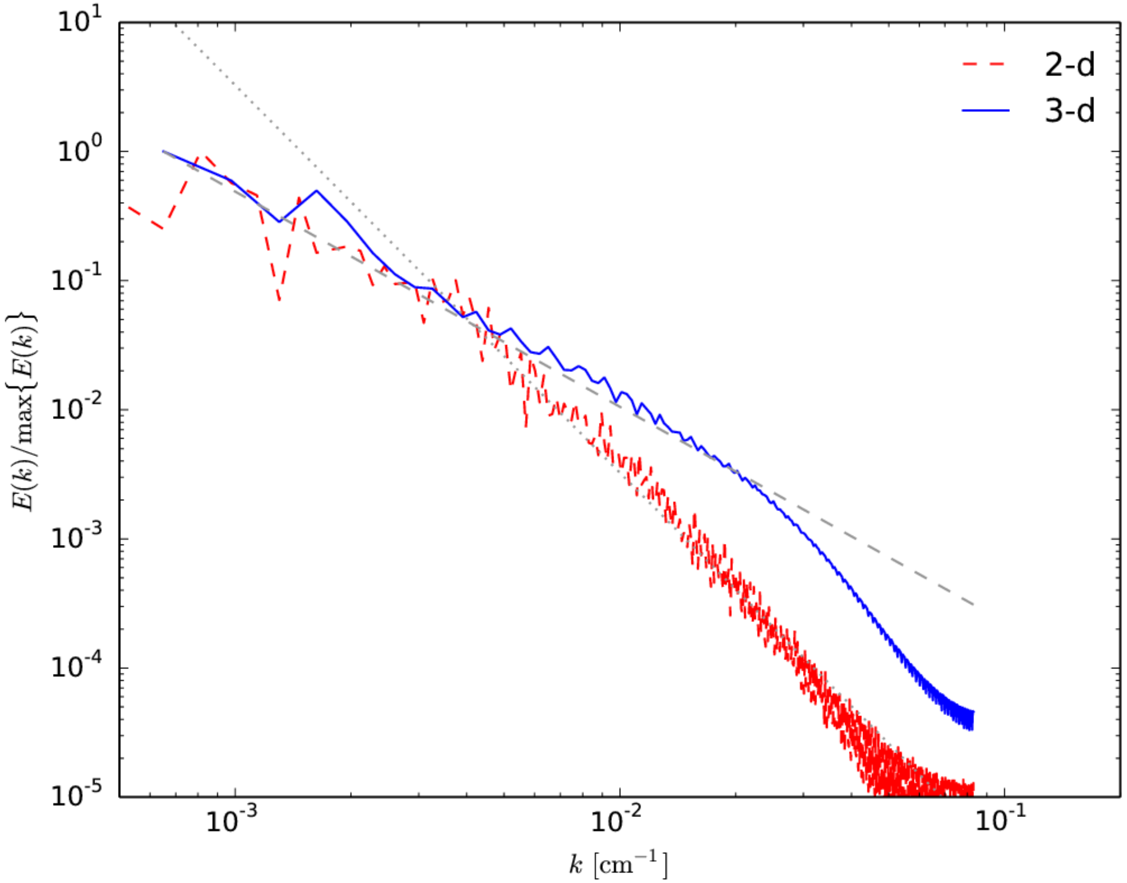
- Real stars are time-dependent
- We can use the stellar structure ODEs if we are in:
  - Hydrostatic equilibrium: dynamic time is fast compared to other timescales
  - Thermal equilibrium: Kelvin-Helmholtz timescale is fast compared to nuclear timescales
- For slow evolution, you can solve for structure, react a bit, solve for new structure, …
- If those timescales matter, then we need to include the time derivatives—equations become PDEs
  - Solution methods are more complex
- Later we'll look a bit at MESA: http://mesa.sourceforge.net/

Beyond 1-d…

- Real stars are three-dimensional
- Convection, turbulence, rotation, instabilities, binary interactions, … are all inherently 3-d phenomena
- However, 3-d is very computationally expensive
  - We use 3-d simulations in stellar evolution to help guide the physics that 1-d codes provide
  - For some explosive events, full 3-d is the only method to do things
- 2-d may sound like a good compromise
  - Less expensive
  - But 2-d behaves very differently than 3-d
- Want to get some experience:
  - Pyro https://python-hydro.github.io/pyro2/ provides implementations of several solvers that you can play with
  - Comprehensive set of notes describe the methods

We need 3-d

- Convection requires 3-d
- Turbulence and instabilities are only properly realized in 3-d
- Core convection requires full 4π



turbulent kinetic energy spectrum in Maestro XRB calculations

Multiscale Problem

- Largest scale: the star itself
- Smallest scales of interest:
  - Dissipation scale?
  - Conductive scale?
  - Reaction zone thickness?
- We are not going to be able to resolve all the scales
  - Subgrid models / ILES

Temporal Challenges

- Many astrophysical explosions exhibit a range of relevant timescales
  - Stellar evolution up to point of explosion / remnant formation ~ millions to 10s of billions of years
  - Simmering convective phase ~ millenia to days/hours
  - Explosion ~ seconds to hours
  - Radiation transport ~ weeks to months
- No single algorithm can model a star from start to finish

- Convective timescale ≫ reaction timescale
- Courant time ≪ stellar evolution timescale
- Standard approach:
- use 1-d stellar evolution code (yea MESA!) to do long term evolution
- model "snapshots" in 3-d
- "inform" the 1-d calculations

Multiphysics

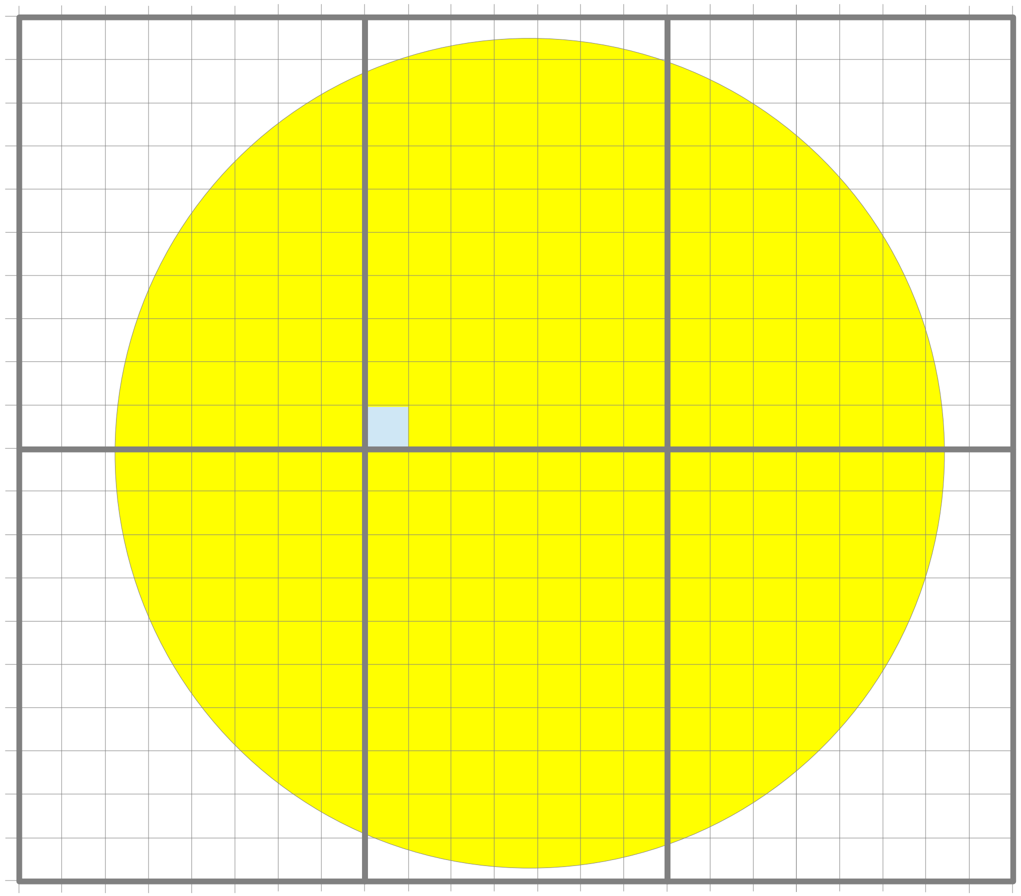
- Stars involve:
  - Hydrodynamics (including turbulence and instabilities)
  - Combustion
  - Self-gravity
  - Radiation / diffusion
  - Magnetic fields
- Different physical processes with different character
  - hyperbolic, elliptic, parabolic
- Range of timescales
  - Timestep restricted by stiffest system
  - Inefficient to just discretize in space and use a single timestep

Astrophysical Approximations

- 1-d stellar evolution codes
  - Still the workhorse for understanding stellar evolution and the stages leading up to explosion
  - Parameter-rich (what does rotation, convection, or turbulence mean in 1-d?)
- Low speed hydrodynamics approximations
  - Developed for atmospheric flows initially
- Compressible (magneto-)hydrodynamics
  - Viscous scales are usually not resolvable
- (Multigroup) Flux limited diffusion radiation hydrodynamics
  - Full multi-angle discretization of radiation can be prohibitively expensive

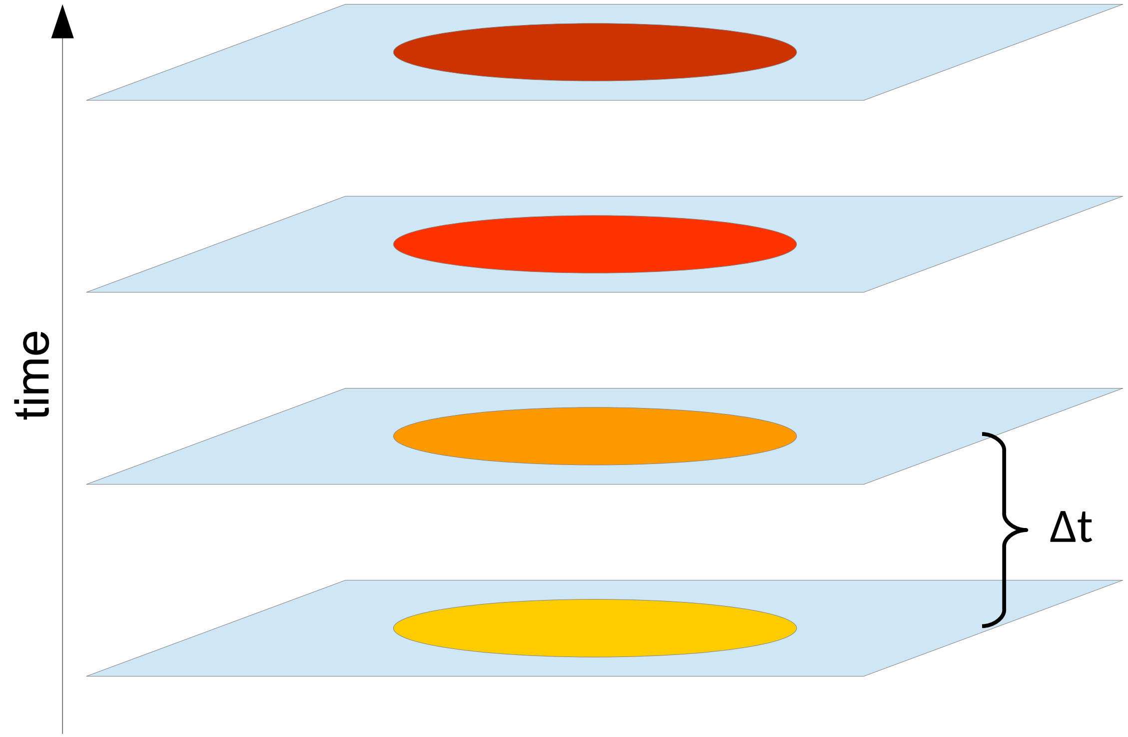
Multidimensional Hydrodynamics

- Star divided into a number of small volumes (zones)
  - Each zone stores average density, velocity, pressure, …
- Conservation laws tell us how the state evolves
  - Fluxes through the faces based on information from neighboring zones
- Differencing transforms PDEs into a system of coupled algebraic equations



Multidimensional Hydrodynamics

- We advance the state in time a little bit (Δt)
  - Lots of steps needed to evolve to see interesting dynamics develop
- Some simulations require 100,000 to a million steps



Multi-d Convection in Stars

- Asymmetries in massive star evolution leading up to core collapse

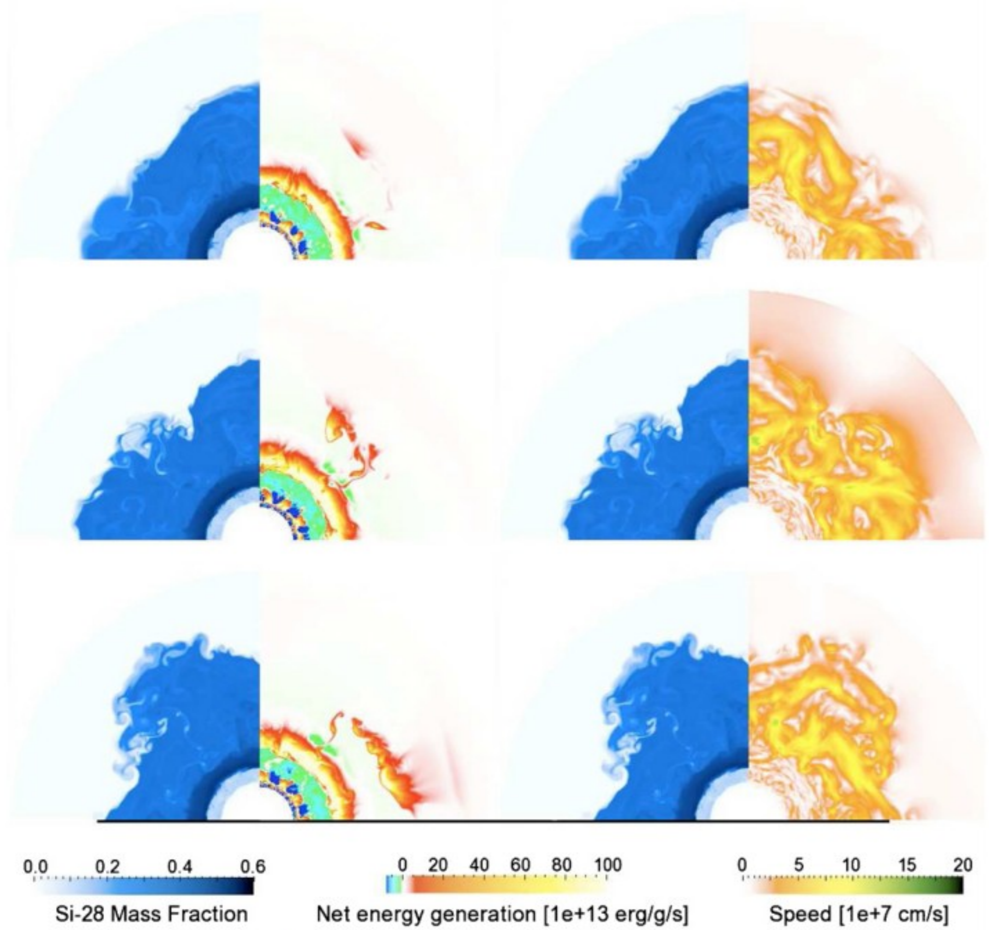


Figure 3. Snapshots of the structure of C, Ne, O, and Si shells surrounding the Fe-core of a pre-collapse progenitor of 23 $M_\odot$ star. Three different times are shown, $t_f = 0$, 61, and 83 s (from top, 0 s, to bottom, 83 s) after our fiducial model (see the text). The left panels (blue) show abundance of Si^{28}, while the right panels show energy generation rate and convective speed, respectively.

(Arnett & Meakin 2011)

Multi-d Convection in Stars

- Exploration of the mixing of protons into C-rich He burning shell

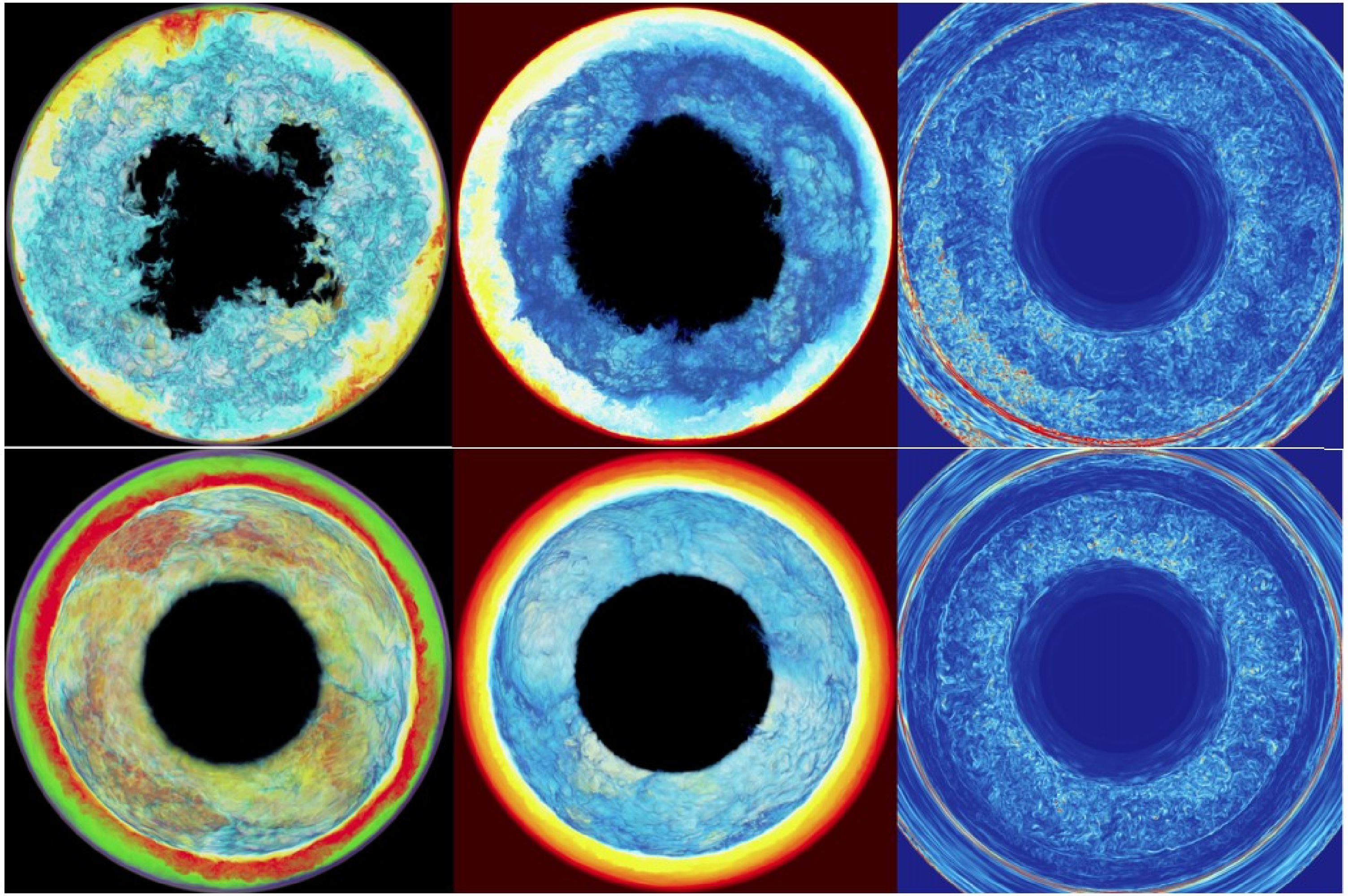


Figure 2. Fractional volume of fluid H+He from 768^3 (left) and 1152^3 (middle) grid runs and vorticity (right) from 1152^3 run at 445min (top row) and 545min (bottom row). The color scale in the fractional volume images maps concentration between 10^{-3} and 10^{-7} and is not exactly identical between left and middle column.

(Herwig et al. 2013)

Common Envelope Evolution

- How does the interaction with a binary companion result in the loss of the stellar envelope?

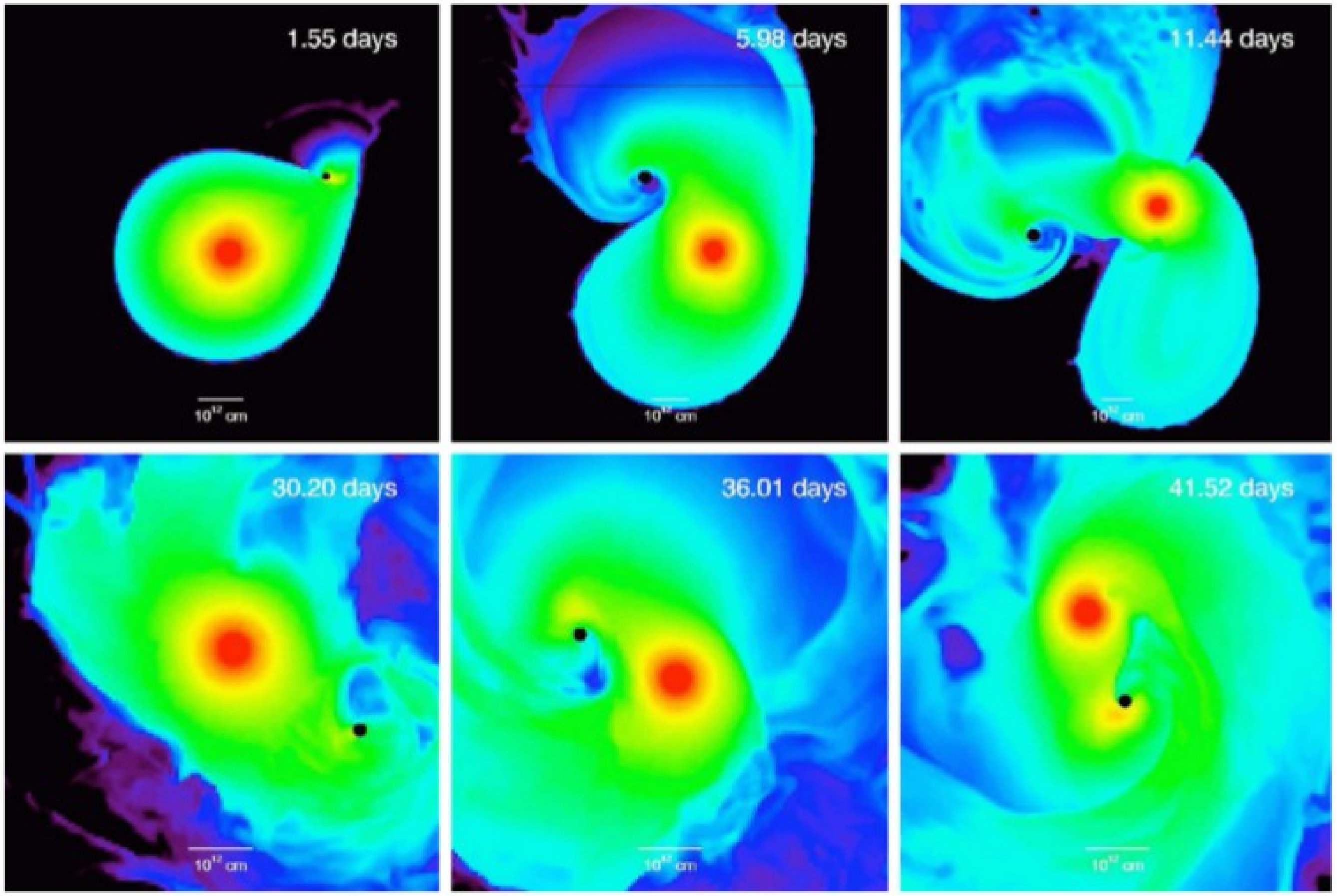


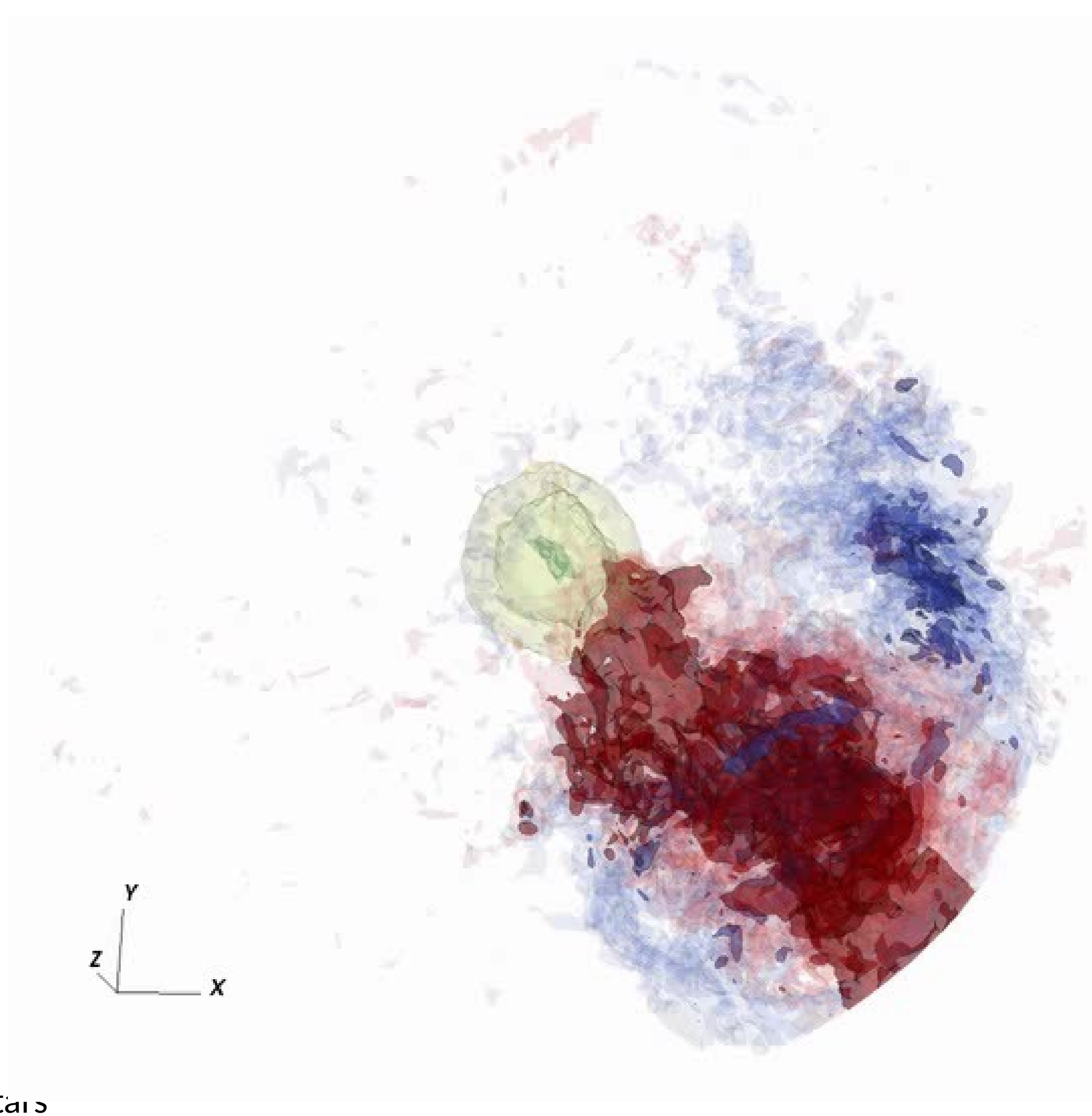
Fig. 10. The density distribution in the orbital plane during the initial and spiral in phase of a binary system composed of a red giant branch star of 1 $M_\odot$ with a 0.7 $M_\odot$ main sequence companion for a system with an initial orbital period of 1 month. Each of the six panels provides the evolution time, which ranges from 1.55 days to 41.52 days. For convenience, a scale of 10^{12} cm is indicated. The black dot indicates the position of the red giant companion.

(Taam & Ricker)

(Taam & Ricker)

Convection Leading Up to SN Ia

- Near-Chandra mass WD has C burning near core…



Some Mult-d Stellar Hydro Applications

- Pre-CC SNe evolution:
  - Arnett & Meakin 2011 (ProMPI); Couch et a. 2015 (Flash); Gilkis & Soker 2015 (Maestro)
- Core He flash:
  - Mocak et al 2008 (Herakles -- prometheus based)
- He shell flash:
  - Herwig et al. 2011 (PPM), Woodward et al. 2015 (PPM)
- Convective Urca
  - Stein & Wheeler 2006 (implicit vulcan?)
- Core H burning
  - Kuhlen et al. 2003 (anelastic); Browning et al. 2004 (A stars + MHD); Meakin & Arnett 2007 (ProMPI); Gilet et al. 2013 (Maestro)
- Convective envelopes
  - Porter et al. 2000 (PPM)
- O Shell burning
  - Lots of Meakin & Arnett 2007 (ProMPI); Kuhlen et al. 2003 (anelastic)
- Pre SNe Ia convection
  - Hoflich & Stein 2002 (implicit); Kuhlen et al. 2006 (anelastic); We've done Maestro models of Chandra & sub-Chandra