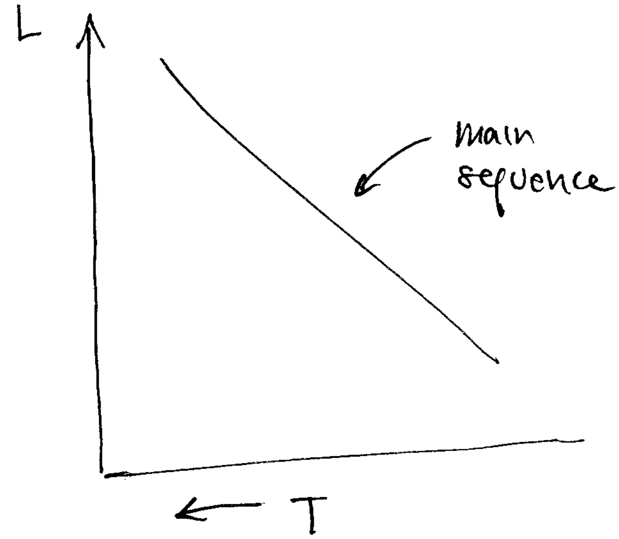
9.

HR diagram



the main sequence has the form

$$\log L = \alpha \log T_{eff} + \text{constant}$$

α changes slightly over the main sequence

We also already expect $L \propto M^{\nu}$

Can we infer this from stellar structure?

- assuming radiative equilibrium
- assuming constant κ (works best for high mass)
- uniform composition μ ← zero age main sequence (ZAMS)
- ideal gas

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$$

$$\frac{dT}{dm} = -\frac{3}{4ac}\frac{\kappa}{T^3}\frac{F}{(4\pi r^2)^2}$$

$$\frac{dF}{dm} = q_0 \rho T^n$$

$$P = \frac{k_B}{\mu m_u} \rho T$$

making these dimensionless requires

$$P_* \sim \frac{GM_*^2}{R_*^4} \quad \text{(i)}$$

$$\rho_* \sim \frac{M_*}{R_*^3} \quad \text{(ii)}$$

$$L_* \sim \frac{ac}{\kappa}\frac{T_*^4 R_*^4}{M_*} \quad \text{(iii)}$$

$$L_* \sim q_0 \rho_* T_*^n M_* \quad \text{(iv)}$$

$$P_* \sim \frac{k_B}{\mu m_u} \rho_* T_* \quad \text{(v)}$$

We can solve these to find scaling relations

starting w/ (v),

$$T_* \approx \frac{\mu m_0}{k_B} \frac{P_*}{\rho_*} \sim \frac{\mu m_0}{k_B} \frac{G M_*^2}{R_*^4} \frac{R_*^3}{M_*}$$

$$\sim \frac{\mu m_0}{k_B} G \frac{M_*}{R_*}$$

putting this into (iii)

$$L_* \sim \frac{ac}{\kappa} \frac{R_*^4}{M_*} \left(\frac{\mu m_0}{k_B} G \frac{M_*}{R_*} \right)^4$$

$$\sim \frac{ac}{\kappa} \left(\frac{G \mu m_0}{k_B} \right)^4 M_*^3$$

looking just at stellar properties

$$L_* \propto \frac{M_*^3 \mu^4}{\kappa}$$

this works over a wide range of the main sequence

We can see the radius dependence, combining our $L_\star$ expression with (iv)

$$L_\star \sim \frac{M_\star^{\cancel{3}\,2}\mu^4}{\kappa} \sim q_0 \rho_\star T_\star^n \cancel{M_\star}$$

inserting for ρ, T, we have

$$L_\star \sim \frac{M_\star^2 \mu^4}{\kappa} \sim q_0 \left(\frac{M_\star}{R_\star^3}\right)\left(\frac{\mu M_\star}{R_\star}\right)^n$$

(dropping κ and q_0)

$$\mu^4 M_\star \sim R_\star^{-3} \mu^n M_\star^n R_\star^{-n}$$

$$\mu^{4-n} M_\star^{1-n} \sim R_\star^{-3-n}$$

$$R_\star \sim \left(\mu^{4-n} M_\star^{1-n}\right)^{1/(-3-n)}$$

$$\therefore \quad R_\star \sim \mu^{(n-4)/(n+3)} M_\star^{(n-1)/(n+3)}$$

Notice that for pp, $n \sim 4$

$$R_\star \sim M_\star^{3/7} \quad \Big\} \text{ low mass}$$

and for CNO, $n \sim 16$

$$R_\star \sim \mu^{12/19} M_\star^{15/19} \quad (\text{almost } R_\star \sim M_\star) \quad \Big\} \text{ massive stars}$$

[In both cases $R_\star$ increases w/ $M_\star$ (unlike WDs!)]

What about density?

$$\rho_\star \sim \frac{M_\star}{R_\star^3} \sim M_\star \left(M^{(n-1)/(n+3)} \right)^{-3}$$

$$\sim M_\star \, M_\star^{(-3n+3)/(n+3)}$$

$$\sim M_\star^{1 + \frac{-3n+3}{n+3}}$$

$$\sim M_\star^{(6-2n)/(n+3)}$$

Notice that for $n > 3$, $\rho_\star$ decreases w/ $M_\star$

— low mass stars are denser than high mass stars

What about the main sequence?

$$L_* \sim R_*^2 \, T_{eff}^4$$

We can eliminate radius via

$$L_* \sim M_*^3$$

$$R_* \sim M_*^{(n-1)/(n+3)} \sim L_*^{\frac{n-1}{3(n+3)}}$$

then

$$L_* \sim L_*^{2(n-1)/3(n+3)} \, T_{eff}^4$$

$$L_*^{1-\frac{2(n-1)}{3(n+3)}} \sim T_{eff}^4$$

for $n = 4$, $\quad L_*^{5/7} \sim T_{eff}^4$

$$\log L_* \sim \frac{28}{5} \log T_{eff} + \text{const} \quad \text{in HR}$$

for $n = 16$,

$$L_*^{9/19} \sim T_{eff}^4$$

$$\log L_* \sim \frac{76}{9} \log T_{eff} + \text{const}$$

$\Longrightarrow$ steeper

15.

Main-sequence lifetime:

$$\tau_{MS} \sim \frac{E}{L} \sim \frac{M}{M^3} \sim M^{-2}$$

since energy reserves are proportional to mass

What is minimum mass for igniting H?

$$T_* \sim \frac{M_*}{R_*} \sim M_* \left(M_*^{\frac{1-n}{n+3}} \right)$$

$$\sim M_*^{1+\frac{1-n}{n+3}} = M_*^{4/n+3}$$

from pp, $n \sim 4$, so $T_* \sim M_*^{4/7}$

compare to sun

$$\frac{T_c}{T_{c,\odot}} = \left(\frac{M}{M_\odot}\right)^{4/7}$$

H can't burn below $\sim 4\times10^6$ K, so we find

$$M_{min} \sim \left(\frac{4\times10^6\,K}{1.5\times10^7\,K}\right)^{7/4} M_\odot \sim 0.1\,M_\odot$$

$$L_{min} \sim \left(\frac{M_{min}}{M_\odot}\right)^3 L_\odot \sim \underline{\underline{10^{-3}\,L_\odot}}$$

lower end of MS