Ch 3: Equations of state

- Main assumption: local thermodynamic equilibrium
  - we already discussed this: at every point in the star we can describe the matter and radiation via a single T

Consider photon mean free path

$$\lambda_\gamma = \underbrace{\frac{1}{n\sigma}}_{\text{this describes the absorbers}} \equiv \frac{1}{\kappa\rho}$$

κ: opacity [cm^2/g]

[Note: some books call $n\sigma = \kappa$ (i.e. no ρ)]

The opacity will contain a lot of atomic physics and details of scattering

We'll see later that electron scattering dominates w/

$$\kappa \sim 1\,cm^2/g$$

Then for $\rho \sim \langle\rho_\odot\rangle \sim 1\,g\,cm^{-3}$, we have $\lambda_\gamma \sim 1\,cm$

[aside: think about how long it will take a photon to random walk from the interior to the surface of the Sun!]

The natural thing to compare to is the pressure scale height

For an isothermal ideal gas, we see from HSE that

$$P = P_0 e^{-x/H} \qquad \text{(your HW)}$$

the scale height is

$$H^{-1} = -\frac{d\ln P}{dx} = \left(\frac{P}{\rho g}\right)^{-1} \qquad \text{(using HSE)}$$

HKT uses λ_P

$$\therefore H = \frac{P}{\rho g}$$

In the Sun, consider the constant density model we derived earlier

$$P = P_c\left[1 - (r/R)^2\right]$$

$$\frac{d\ln P}{dr} = \frac{d}{dr}\left[\ln P_c + \ln\left[1 - (r/R)^2\right]\right]$$

$$= -\frac{1}{1-(r/R)^2}\, 2\frac{r}{R^2}$$

$$\therefore H = -\left(\frac{d\ln P}{dr}\right)^{-1} = \frac{R^2}{2r}\left[1 - (r/R)^2\right]$$

consider $H(R/2) = \frac{R^2}{R}\left[1 - (1/2)^2\right] = \frac{3}{4}R$

so $H \sim R \gg \lambda_\gamma$

This supports our statement that we are in LTE

Recall a distribution function

$$n(p)\, d^3x\, d^3p = \#\text{ of particles w/ momentum } p \text{ in volume } d^3x$$

So far we've been using n as number density (cm^{-3})

We'll introduce $\tilde{N}$ as specific number density (ugly notation)

$$\tilde{N} \equiv n/\rho$$

Then we can define the chemical potential as

$$\mu_i \equiv \left.\frac{\partial e}{\partial \tilde{N}}\right|_{s,v}$$ here e is specific internal energy, HKT uses E

chemical equilibrium says that

$$\sum_i \mu_i\, d\tilde{N}_i = 0$$ (where N_i can change due to reactions)

Not all $d\tilde{N}_i$ are independent — usually when one goes up, another goes down

Note: it can be argued that $\mu_\gamma = 0$ (photons), since their # is not conserved

We'll use this later when we deal w/ ionization...

Our form of the first law is

$$T\,ds = de + p\,d(1/\rho) - \mu\,d\tilde{N}$$

these are all specific quantities

s is [erg/g/K]

e is [erg/g]

p is [erg/cm^3] (normal pressure units]

$1/\rho$ is [cm^3/g] – specific volume

$\tilde{N}$ is [1/g] – specific number density

μ is [erg]

Note: $\mu = \left.\frac{\partial e}{\partial \tilde{N}}\right|_{s,v}$ – this follows from the first law

Why is μ negative for an ideal gas?

if s is constant, adding a particle brings in μ more energy, so e needs to decrease (at constant ρ) for s to remain constant

see "Understanding the Chemical Potential"

General distribution function

$$n(p) = \underbrace{\frac{1}{h^3}}_{\substack{\text{phase space} \\ \text{momentum} \\ \text{volume}}} \frac{g}{e^{(-\mu + \underbrace{\varepsilon_0}_{\text{reference energy}} + \varepsilon(p))/kT} \underbrace{\pm}_{} 1}$$

g ← degeneracy of states w/ ε_0

+ for Fermions ($\frac{1}{2}$ spin)

− for Bosons (0 or integer spin)

From this, the number density is simply

$$n = \underbrace{\int_p}_{\substack{\text{integrate} \\ \text{over all of} \\ \text{momentum} \\ \text{space}}} n(p) \underbrace{4\pi p^2 \, dp}_{\substack{\text{integration} \\ \text{over a sphere}}} \qquad (\text{cm}^{-3})$$

generally, $\varepsilon(p) = (p^2c^2 + m^2c^4)^{1/2} - \underbrace{mc^2}_{\substack{\text{rest} \\ \text{mass}}}$ ← this is kinetic energy

for $pc \ll mc^2$

$$\varepsilon(p) = mc^2\left(\frac{p^2c^2}{m^2c^4} + 1\right)^{1/2} - mc^2$$

$$\sim mc^2\left(1 + \frac{1}{2}\frac{p^2c^2}{m^2c^4}\right) - mc^2 = \frac{1}{2}\frac{p^2}{m}$$ (non-relativistic expression)

for $pc \gg mc^2$

$$\varepsilon(p) = pc\left(1 + \frac{m^2c^4}{p^2c^2}\right)^{1/2} - mc^2 \sim pc\left(1 + \frac{1}{2}\frac{m^2c^4}{p^2c^2}\right) - mc^2$$

$$\sim pc$$

We'll need velocity from Hamiltonian mechanics

$$v = \frac{\partial \mathcal{E}}{\partial p}$$

Then pressure is

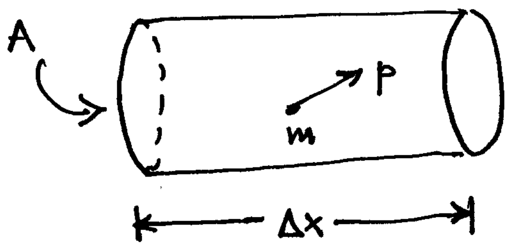
$$P = \frac{1}{3} \int_p n(p)\, p\, v\, 4\pi p^2\, dp$$

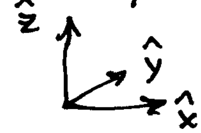
and internal energy is

$$\rho e = \int_p n(p)\, \mathcal{E}(p)\, 4\pi p^2\, dp$$

Motivation of the p integral (see e.g. C&O §10.2)

consider a cylinder w/ gas which elastically collides. When colliding w/ an end, Δp is entirely in $\hat{x}$





change in momentum: $\Delta p = -2 p_x \hat{x}$

Now the force on the wall is just

$$F_{wall} = -\frac{\Delta p}{\Delta t} = \frac{2 p_x}{\Delta t} \hat{x}$$

equal and opposite

The time between collisions (to the other end and back) is

$$\Delta t = 2 \frac{\Delta x}{v_x}$$

so $$F_{wall} = \frac{p_x v_x}{\Delta x} \hat{x}$$

if we average over particles, we need

$$\langle p_x v_x \rangle = \langle p_y v_y \rangle = \langle p_z v_z \rangle = \frac{1}{3} p \cdot v$$

no preferred direction

$\therefore$ the average force / particle is $f(p) = \frac{1}{3} \frac{pv}{\Delta x}$

if $N(p)d^3p$ is the distribution function integrated over space (so just # of particles w/ p in $p, p+\Delta p$)

then the total force from all particles w/ momentum p is

$$dF(p) = f(p)\,N(p)\,d^3p = \frac{1}{3}\frac{N(p)}{\Delta x}\,p v\, d^3p$$

if we divide by the area of the wall, A, we get a pressure

$$dP = \frac{dF(p)}{A} = \frac{1}{3}\frac{N(p)}{\Delta V}\,p v\, d^3p$$

but since $n(p)d^3p$ is the # of particles /volume w/ momentum p, we have

$$n(p)d^3p = \frac{N(p)}{\Delta V}\,d^3p$$

so

$$dP = \frac{1}{3}\,n(p)\,d^3p\;p v$$

integrating over all momenta,

$$P = \frac{1}{3}\int_0^\infty n(p)\,p\,v\,4\pi p^2\,dp$$

Photon gas

two independent states (polarizations)

$$n(p) = \frac{2}{h^3} \frac{1}{e^{pc/kT} - 1}$$

only one energy level - $\varepsilon_j = 0$, and $\mu_\gamma = 0$

we can derive the γ # density — note: we should think of this as an average resulting from interactions w/ matter

$$n_\gamma = 4\pi \int_0^\infty n(p)\, p^2\, dp = \frac{8\pi}{h^3} \int_0^\infty \frac{p^2}{e^{pc/kT} - 1}\, dp$$

take $x = \frac{pc}{kT} \longrightarrow dx = \frac{c}{kT}\, dp$

$$= \frac{8\pi}{h^3} \left(\frac{kT}{c}\right)^3 \underbrace{\int_0^\infty \frac{x^2}{e^x - 1}\, dx}_{= 2\zeta(3) \sim 2(1.202\ldots)}$$

zeta function

$$\therefore\ n_\gamma = 2\pi \zeta(3) \left(\frac{2kT}{hc}\right)^3 \sim 20.3\, T^3\ \mathrm{cm}^{-3}$$

Radiation pressure is

$$P_\gamma = \frac{1}{3}\int_p 4\pi p^2\, p v\, n(p)\, dp$$

now $v = c$ and $E = h\nu = pc \longrightarrow p = \frac{h\nu}{c}$

$$= \frac{4\pi}{3}\int_\nu c\left(\frac{h\nu}{c}\right)^3\left[\frac{2}{h^3}\,\frac{1}{e^{h\nu/kT}-1}\right]\frac{h}{c}\,d\nu$$

$$= \frac{8\pi}{3}\frac{h}{c^3}\underbrace{\int_\nu \frac{\nu^3\, d\nu}{e^{h\nu/kT}-1}}$$

this looks an awful lot like B_ν — the Planck func

taking $x \equiv \frac{h\nu}{kT}$

$$= \frac{8\pi}{3}\frac{h}{c^3}\left(\frac{kT}{h}\right)^4\int\frac{x^3\,dx}{e^x-1} = \frac{8\pi}{3}\frac{(kT)^4}{(ch)^3}\frac{\pi^4}{15} = \frac{1}{3}aT^4$$

a is the radiation constant, $a = 7.56\times10^{-15}$ erg cm^{-3} K^{-4}

finally,

$$pe_\gamma = \int_p 4\pi p^2\, \mathcal{E}(p)\, n(p)\, dp = 3P_\gamma\,, \quad \text{since } \mathcal{E}(p) = pc$$

$\therefore$ this is a $\gamma = \frac{4}{3}$ EOS

The integrand of our energy integral

$$u_\gamma\, d\nu = \frac{8\pi h\nu^3}{c^3}\frac{d\nu}{e^{h\nu/kT}-1} = \frac{4\pi}{c}B_\nu(T)\, d\nu$$

$\therefore$ the γ energy density / unit frequency is just $\propto$ Planck func.

Ideal gas

When is ideal gas valid?

- inside the star, we are ionized
- Coulomb interactions can occur

distance between particles is $d \sim \frac{1}{n^{1/3}}$

recall that $n = \frac{\rho}{A m_u}$

so $d \sim \left(\frac{A m_u}{\rho}\right)^{1/3} \sim \left(\frac{A m_u 4\pi}{3M}\right)^{1/3} R$ (using $\rho = \bar{\rho}$)

Coulomb energy is $\epsilon_c \sim \frac{Z^2 e^2}{d}$ (CGS)

To understand if this is important, we compare to thermal energy, kT

$$\frac{\epsilon_c}{kT} \sim \frac{Z^2 e^2}{d kT}$$

taking our Virial estimate, $kT \sim \frac{1}{3} \frac{GMA m_u}{R}$

$$\frac{\epsilon_c}{kT} \sim \frac{Z^2 e^2}{A^{4/3} m_u^{4/3} G M^{2/3}} \sim 0.01$$ for $Z = A = 1$ and $M = M_\odot$

$\therefore$ we can ignore Coulomb interactions

For an ideal gas, we start w/

$$n(p) = \frac{1}{h^3} \frac{g}{e^{(-\mu + \varepsilon_0 + \varepsilon(p))/kT} \pm 1}$$

we'll start by asserting that $\frac{\mu}{kT} \ll -1$ (we'll check this later)

In this case, $e^{-\mu/kT}$ will dominate over the '± 1' term

so

$$n(p) \sim \frac{g}{h^3} e^{\mu/kT} e^{-\varepsilon_0/kT} e^{-p^2/2mkT}$$

(again, assuming only one energy level)

and then # density is

$$n = \frac{4\pi g}{h^3} \int_0^\infty p^2 e^{\mu/kT} e^{-\varepsilon_0/kT} e^{-p^2/2mkT} \, dp$$

$$= \frac{4\pi g}{h^3} e^{\mu/kT} e^{-\varepsilon_0/kT} \int_0^\infty p^2 e^{-p^2/2mkT} \, dp$$

take $x = \frac{p}{\sqrt{2mkT}}$ $\quad dx = \frac{dp}{\sqrt{2mkT}}$

$$n = \frac{4\pi g}{h^3} e^{\mu/kT} e^{-\varepsilon_0/kT} (2mkT)^{3/2} \underbrace{\int_0^\infty x^2 e^{-x^2} dx}_{= \frac{1}{2}\Gamma\left(\frac{3}{2}\right)}$$

$$\boxed{\Gamma\left(\tfrac{3}{2}\right) = \frac{\sqrt{\pi}}{2}}$$

Chemical potential

$$de = T\underline{ds} - p\,d(\underline{1/\rho}) + \mu\, d\underline{\tilde{N}}$$

In order to add a particle while keeping entropy and volume constant, the particle must contain negative energy, so μ must be negative.

Consider the entropy of an ideal gas

$$S = Nk\left[\log\left(g\frac{V}{N}\right) + \frac{3}{2}\log\left(\frac{mU}{3N\pi\hbar^2}\right) + \frac{5}{2}\right]$$

not specific

If we add a particle, then we have

$$S' = (N+1)k\left[\log\left(g\frac{V}{N+1}\right) + \frac{3}{2}\log\left(\frac{m(U+\mu)}{3(N+1)\pi\hbar^2}\right) + \frac{5}{2}\right]$$

where μ is the energy the additional particle brings and U is unchanged, since we are doing this at constant volume.

requiring $S = S'$ implies $\mu < 0$ (lots of algebra)

In effect: the only way to hold the entropy of the system constant is to require U to decrease, which suppresses the # of microstates

see "Understanding the Chemical Potential" so $$e^{\mu/kT} = \frac{nh^3}{g(2m\pi kT)^{3/2}} e^{\varepsilon_0/kT}$$

our requirement that $e^{\mu/kT} \ll 1$ means $\frac{n}{T^{3/2}}$ needs to be small

if $e^{\mu/kT}$ is not small, corrections will appear — we will not consider those

Ex: for the Sun, $n = \frac{\rho}{\mu m_u}$

at the center, $\rho \sim 150\ g/cm^3$
$\mu \sim 0.6$

Then $$n = \frac{150\ g/cm^3}{0.6 \cdot 1.67\times 10^{-29}\ g} = 1.5\times 10^{26}\ cm^{-3}$$

$T \sim 1.5\times 10^7\ K$

then $$e^{\mu/kT} = \frac{1.5\times 10^{26}\ cm^{-3} \cdot (6.63\times 10^{-27}\ erg\cdot s)^3}{g(2\cdot 1.67\times 10^{-29}\ g \cdot \pi \cdot 1.38\times 10^{-16}\ erg/K \cdot 1.5\times 10^7\ K)^{3/2}} e^{\varepsilon_0/kT}$$

$$\sim 1.4\times 10^{-5}\ e^{\varepsilon_0/kT}$$

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We have

$$e^{\mu/kT} = \frac{n h^3}{g(2m\pi kT)^{3/2}} e^{\varepsilon_0/kT}$$

log of both sides

$$\frac{\mu}{kT} = \log \frac{n}{g} - \log \left(\frac{2m\pi kT}{h^2}\right)^{3/2} + \frac{\varepsilon_0}{kT}$$

$$\frac{\mu}{kT} = -\log \left(\frac{g}{n} \left(\frac{2m\pi kT}{h^2}\right)^{3/2} \right) + \frac{\varepsilon_0}{kT}$$

we take $\varepsilon_0 = 0$ for ideal gas (no reference energy)

then we have

$$\mu = -kT \log \left[\underbrace{\frac{g}{n} \left(\frac{2m\pi kT}{h^2} \right)^{3/2}}_{\text{this is } >1} \right]$$

Now $$P = \frac{4\pi}{3}\int_0^\infty p v\, n(p)\, p^2\, dp$$

take $v = p/m$

$$= \frac{g}{3}\frac{4\pi}{h^3}\frac{1}{m}\int_0^\infty p^4 e^{\mu/kT} e^{-\varepsilon_0/kT} e^{-p^2/2mkT}\, dp$$

take $x = \frac{p}{\sqrt{2mkT}} \rightarrow dx = \frac{dp}{\sqrt{2mkT}}$

$$\therefore \quad P = \frac{g}{3}\frac{4\pi}{h^3}\frac{1}{m} e^{\mu/kT} e^{-\varepsilon_0/kT} (2mkT)^{5/2} \underbrace{\int_0^\infty x^4 e^{-x^2} dx}_{\frac{1}{2}\Gamma(\frac{5}{2})}$$

$$= \frac{g}{\cancel{3}}\frac{4\pi}{h^3}\frac{1}{m} e^{\mu/kT} e^{-\varepsilon_0/kT} (2mkT)^{5/2} \cdot \frac{\cancel{3}}{8}\sqrt{\pi}$$

$$= \frac{1}{2}\cdot\frac{3}{4}\sqrt{\pi}$$

$$= g\,\frac{\pi^{3/2}}{m} (2mkT)^{5/2} e^{\mu/kT} e^{-\varepsilon_0/kT}\, \frac{1}{2}\frac{1}{h^3}$$

substituting in our expression of $e^{\mu/kT}$

$$P = \frac{g}{2}\frac{1}{h^3}\frac{\pi^{3/2}}{m}(2mkT)^{5/2} e^{-\varepsilon_0/kT}\left[\frac{nh^3}{g(2m\pi kT)^{3/2}}\right] e^{\varepsilon_0/kT}$$

$$= nkT$$

The energy is then

$$\rho e = 4\pi \int_p n(p)\,\varepsilon(p)\, p^2\, dp = 4\pi \int n(p)\,\frac{p^2}{2m}\, p^2\, dp = \frac{3}{2} P$$

so the ideal gas is a $\gamma = 5/3$ gas

Fermi EOS

e^-, p, n all have spin $\frac{1}{2}$ $\longrightarrow$ $g=2$

We'll look at electron degeneracy

We take $\mathcal{E}_0 = mc^2$

$$n = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2\,dp}{e^{(-\mu+mc^2+\mathcal{E}(p))/kT}+1}$$

$\uparrow$ fermions

and $\mathcal{E}(p) = mc^2\left[\sqrt{1+\left(\frac{p}{mc}\right)^2} - 1\right]$

$$v = \frac{\partial \mathcal{E}}{\partial p} = \frac{1}{\cancel{2}}\cancel{mc^2}\left(1+\left(\frac{p}{mc}\right)^2\right)^{-\frac{1}{2}}\frac{\cancel{2}p}{m\cancel{c}}\frac{1}{\cancel{mc}}$$

$$= \frac{p}{m}\left(1+\left(\frac{p}{mc}\right)^2\right)^{-\frac{1}{2}}$$

$$\left(\sim \frac{p}{m}\left[1-\frac{1}{2}\left(\frac{p}{mc}\right)^2\right] \text{ for } p \ll mc\right)$$

Doing the general integral is hard. Lots of approximations exist. We won't delve into all of those, but consider:

complete degeneracy

$$F(\mathcal{E}) \equiv \frac{1}{e^{(\mathcal{E}-(\mu-mc^2))/kT}+1}$$

consider $T \longrightarrow 0$

if $\mathcal{E} > (\mu - mc^2)$ then we get $\frac{1}{e^{+\infty}+1} \sim 0$

if $\mathcal{E} < (\mu - mc^2)$ then we get $\frac{1}{e^{-\infty}+1} \sim 1$

so the distribution function becomes a step function

Fermi energy: $\mathcal{E}_F \equiv \mu - mc^2$

$$F(\mathcal{E}) = \begin{cases} 1 & \mathcal{E} < \mathcal{E}_F \\ 0 & \mathcal{E} > \mathcal{E}_F \end{cases} \qquad n(p) = \frac{2}{h^3} F(p)$$

All states w/ $\mathcal{E} < \mathcal{E}_F$ are completely filled

Fermi momentum: $x_F = \frac{p_F}{mc}$ then $\mathcal{E}_F = mc^2\left[(1+x_F^2)^{1/2} - 1\right]$

Note that $\mu_F = \mathcal{E}_F + mc^2$ is the total ~~mass~~ energy of the most energetic particle in the system

How do we find ε_F, x_F, or μ_F?

We constrain things based on the # density &

$$n = \underbrace{\frac{8\pi}{h^3}\int_0^{p_F} p^2\, dp}_{\substack{\text{we only need to}\\ \text{integrate to } p_F,\\ \text{since } F(p)=0\\ \text{beyond that}}} = 8\pi\left(\frac{mc}{h}\right)^3 \int_0^{x_F} x^2\, dx = \frac{8\pi}{3}\underbrace{\left(\frac{mc}{h}\right)}_{1/\lambda}^3 x_F^3$$

where λ is the Compton wavelength

$$\therefore\ x_F = \left(\frac{3}{8\pi}\right)^{1/3} \frac{h}{mc}\, n^{1/3}$$

Now $n_e = \dfrac{\rho}{\mu_e m_u}$ for electrons, and

$$n_e = \frac{8\pi}{3}\left(\frac{mc}{h}\right)^3 x_F^3 = \frac{\rho}{\mu_e m_u}$$

$$\frac{\rho}{\mu_e} = \frac{8\pi m_u}{3}\left(\frac{mc}{h}\right)^3 x^3 \sim 10^6\ g/cm^3\ x^3$$

Note that $x = x_F \sim 1$ is about the dividing line between non-relativistic and extremely relativistic

In a WD, the above shows this occurs @ $\rho/\mu_e \sim 10^6\ g/cm^3$

(for NS, $m_e \rightarrow m_n$ which is $\sim 10^3 \times$ larger, so $\rho \sim 10^9 \times$ larger)

Also note: there is no T dependency — later we'll look at some #s to see how a finite T changes things.

What about P?

$$P = \frac{1}{3}\int p v\, n(p)\, dp$$

$$= \frac{4\pi}{3}\underbrace{\frac{1}{h^3}2}_{g}\frac{1}{m}\int_0^{p_F}\frac{p^4}{\left(1+\left(\frac{p}{mc}\right)^2\right)^{1/2}}\,dp$$

$$x = \frac{p}{mc}$$

$$= \frac{8\pi}{3}\frac{1}{m}\frac{1}{h^3}(mc)^5\int_0^{x_F}\frac{x^4}{\sqrt{1+x^2}}\,dx \equiv A\, f(x)$$

$$A = \frac{\pi}{3}\left(\frac{mc}{h}\right)^3 mc^2$$

$$f(x) = x\,(2x^2-3)(1+x^2)^{1/2} + 3\sinh^{-1}x$$

Note to use this, you first find $x = x_F$ from the # density expression, and then evaluate P w/ that x

Aside: to do this integral

$$I \equiv \int_0^{x_F} \frac{x^4}{(1+x^2)^{1/2}}\, dx$$

$$u = (1+x^2)^{1/2}$$

$$u^2 = (1+x^2)$$

$$2u\,du = 2x\,dx \longrightarrow dx = \frac{u\,du}{x} = \frac{u\,du}{(u^2-1)^{1/2}}$$

$$I = \int_1^{(1+x_F^2)^{1/2}} \frac{(u^2-1)^2}{u} \frac{u\,du}{(u^2-1)^{1/2}} = \int_1^{(1+x_F^2)^{1/2}} (u^2-1)^{3/2}\, du$$

$$= \frac{u(u^2-1)^{3/2}}{4} - \frac{3u(u^2-1)^{1/2}}{8} + \frac{3}{8}\ln\left(u + (u^2-1)^{1/2}\right)$$

Spiegel 12.10.21

$$= \frac{1}{8}\left[2(1+x_F^2)^{1/2} x_F^3 - 3(1+x_F^2)^{1/2} x_F + 3\ln \underbrace{\left[(1+x_F^2)^{1/2} + x_F\right]}_{= \sinh^{-1} x_F}\right]$$

(Spiegel 14.55)

$$\therefore\ I = \frac{1}{8}\left[x_F(2x_F^2 - 3)(1+x_F^2)^{1/2} + 3\sinh^{-1} x_F\right]$$

$$= \frac{1}{8} f(x)$$

E is similar

$$E = \int n(p)\, \mathcal{E}(p)\, 4\pi p^2\, dp$$

$$= 4\pi \frac{2}{h^3} \int_0^{p_F} p^2\, mc^2 \left[\left(1 + \left(\frac{p}{mc}\right)^2\right)^{1/2} - 1 \right] dp$$

$$x = \frac{p}{mc}$$

$$= \frac{8\pi}{h^3}\, mc^2\, (mc)^3 \int_0^{x_F} x^2 \left[(1+x^2)^{1/2} - 1 \right] dx$$

$$= A\, g(x)$$

same A, $\quad A = \frac{\pi}{3} \left(\frac{mc}{h}\right)^3 mc^2$

$$g(x) = 8x^3 \left[(1+x^2)^{1/2} - 1 \right] - f(x)$$

note: this E is erg/cm^3 (same as dyn/cm^2)

Remember: this is in the limit of $T \to 0$

but we did not make any approx
as to whether we are relativistic
or non-relativistic ... yet.

In future hmwk, we'll consider the expansions of $f(x)$ and $g(x)$ in the limits that $x \ll 1$ and $x \gg 1$

We find

$$P_e = \begin{cases} A \frac{8}{5} x^5 & x \ll 1 \\ A\, 2x^4 & x \gg 1 \end{cases}$$

taking $x = \left(\frac{3}{8\pi}\right)^{1/3} \left(\frac{h}{mc}\right) n_e^{1/3} = \left(\frac{3}{8\pi}\right)^{1/3} \left(\frac{h}{mc}\right) \left(\frac{\rho}{\mu_e m_u}\right)^{1/3}$

we have

$$P_e \propto \begin{cases} \left(\rho/\mu_e\right)^{5/3} & \text{non-relativistic} \\ \left(\rho/\mu_e\right)^{4/3} & \text{relativistic} \end{cases}$$

and the same ρ-dependence for E

Also we can show

$$\frac{\rho e_e}{P_e} = \begin{cases} 3/2 & (\gamma = 5/3) & x \ll 1 \\ 3 & (\gamma = 4/3) & x \gg 1 \end{cases}$$

WD structure!

Consider a WD — degenerate electrons provide most of the support

Virial theorem:

$$3(\gamma-1)\,U = -\Omega$$

assume constant ρ

$$\Omega = -\frac{3}{5}\frac{GM_\star^2}{R_\star}$$

$$U = V E_e = \left(\frac{4\pi}{3}R_\star^3\right)\overbrace{\left(\frac{12}{5}A x^5\right)}^{\frac{3}{2}P \text{ in NR limit}}$$

take $\dfrac{\rho}{\mu_e} = Bx^3 \qquad B = \dfrac{8\pi m_u}{3}\left(\dfrac{m_e c}{h}\right)^3$

$$x = \left(\frac{1}{B}\frac{\rho}{\mu_e}\right)^{1/3} = \left(\frac{1}{B}\frac{3}{4\pi}\frac{M_\star}{R_\star^3}\right)^{1/3}$$

then, w/ $\gamma = \frac{5}{3}$,

$$U = -\frac{1}{2}\Omega$$

$$\frac{4\pi}{\cancel{3}}R_\star^3\,\frac{\overset{4}{\cancel{12}}}{\cancel{5}}A\left(\frac{1}{B}\frac{3}{4\pi}\frac{M_\star}{R_\star^3}\right)^{5/3} = \frac{\cancel{3}}{\underset{2}{\cancel{10}}}\frac{GM_\star^2}{R_\star}$$

$$\frac{32\pi}{3}\left(\frac{3}{4\pi}\right)^{5/3}\frac{A}{G B^{5/3}}\,\cancel{M_*^{5/3}}\,\frac{1}{R_*^{\cancel{2}}} = \frac{M_*^{\cancel{2}\ 1/3}}{\cancel{R_*}}$$

$$M_*^{1/3} \sim \frac{1}{R_*} \quad \text{(for constant density)}$$

This means that more massive WDs are smaller

Putting in #s finds

$$\frac{M_*}{M_\odot} \sim 10^{-6}\left(\frac{R}{R_\odot}\right)^{-3}\left(\frac{2}{\mu_e}\right)^{5}$$

This implies WDs have radii ~ Earth's radius

[Later we'll do better by deriving a second-order ODE for the structure of a polytrope gas

This result is for non-relativistic WDs!

What about the relativistic case?

$$\gamma = 4/3$$

$$U = -\Omega$$

$$\left(\frac{4}{3}\pi R_*^3\right) A\, 6x^4 = \frac{3}{5}\frac{GM_*^2}{R_*}$$

and

$$x = \left(\frac{1}{B}\frac{\rho}{\mu_e}\right)^{1/3}$$

then

$$8\pi R_*^3 A \left(\frac{1}{B}\frac{3M_*}{4\pi R_*^3}\frac{1}{\mu_e}\right)^{4/3} = \frac{3}{5}\frac{GM_*^2}{R_*}$$

notice that R_* cancels out!

We are left with

$$M_*^{2/3} = \frac{40\pi}{3}\frac{A}{G}\left(\frac{3}{4\pi B}\right)^{4/3}\left(\frac{1}{\mu_e}\right)^{4/3} \longrightarrow M_* \sim \text{const}\cdot\left(\frac{1}{\mu_e}\right)^2$$

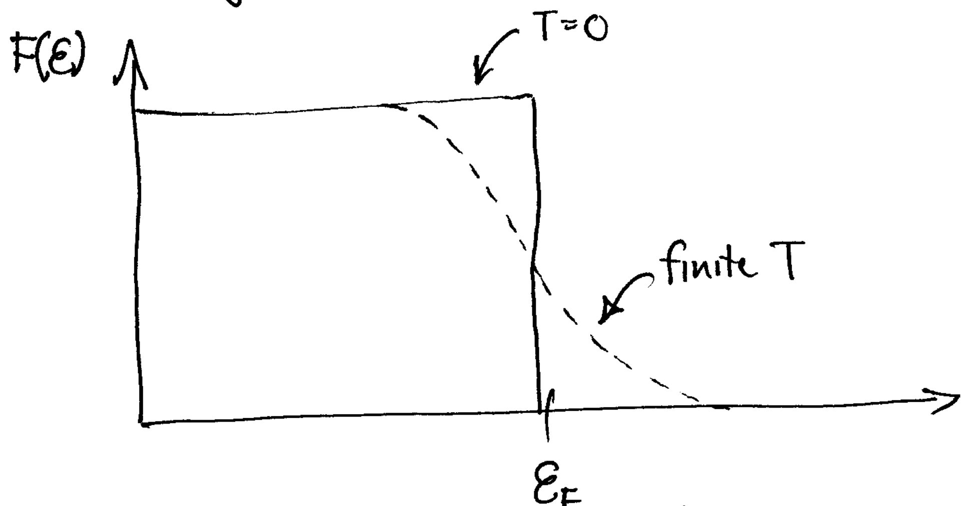
We can evaluate this, but b/c of the approximations we made, it will not be the correct value.

Later we'll see how to solve this self-consistently, and we'll find

$$M_c \sim 1.4\, M_\odot \left(\frac{2}{\mu_e}\right)^2$$

So far, we assumed $T \to 0$ (complete degeneracy)

What about finite T? When does T begin to matter (lift the degeneracy)?



when $T \neq 0$, some electrons can take on energies, $\mathcal{E} > \mathcal{E}_F$

For low T, most of the electrons are still as they were w/ complete degeneracy.

If we measure energy as kT and $kT \ll \mathcal{E}_F$, then only the electrons already near $\mathcal{E}_F$ can escape to the region $\mathcal{E} > \mathcal{E}_F$

We can use $\mathcal{E}_F \sim kT$ as the transition region between degeneracy and non-degeneracy

Since $\mathcal{E}_F \sim \frac{1}{2} mc^2 x_F^2 = \frac{1}{2} mc^2 \left(\frac{1}{B}\frac{\rho}{\mu_0}\right)^{2/3} \sim kT$

(N-R) for degeneracy

$$\therefore \quad \left(\tfrac{1}{2} m_e c^2\right)^{3/2} \frac{1}{B} \frac{\rho}{\mu_e} = k^{3/2} T^{3/2}$$

$$\frac{\rho}{\mu_e} = B \left(\frac{2k}{m_e c^2}\right)^{3/2} T^{3/2} \sim 6 \times 10^{-9} \left(\frac{T}{1\mathrm{K}}\right)^{3/2} \mathrm{g\ cm^{-3}}$$

if $\frac{\rho}{\mu_e} > 6 \times 10^{-9} \left(\frac{T}{1\mathrm{K}}\right)^{3/2} \mathrm{g\ cm^{-3}}$, we are degenerate (N-R)

For the relativistic case,

$$\varepsilon_F \sim mc^2 x_F \sim mc^2 \left(\frac{1}{B} \frac{\rho}{\mu_e}\right)^{1/3} \sim kT$$

$$\therefore \frac{\rho}{\mu_e} = \left(\frac{k}{mc^2}\right)^3 B\, T^3$$

$$= 4.6 \times 10^{-24} \left(\frac{T}{1\mathrm{K}}\right)^3 \mathrm{g\ cm^{-3}}$$

If T is large, partial degeneracy ~~be~~ is attained

Today we usually do the necessary integrals numerically.

What about degenerate ions?

We have

$$P = \frac{\pi}{3}\left(\frac{mc}{h}\right)^3 mc^2 f(x_F)$$

where $f(x) = x(2x^2 - 3)(1+x^2)^{1/2} + 3\sinh^{-1} x$

$$x_F = \frac{p_F}{mc}$$

for non-relativistic

$$P \sim \frac{\pi}{3}\left(\frac{mc}{h}\right)^3 mc^2 \frac{8}{5} x_F^{\,2}$$

From # density, we have

$$x_F = \left(\frac{3}{8\pi}\right)^{1/3}\left(\frac{h}{mc}\right) n^{1/3}$$

$$\therefore\ P \sim \frac{\pi}{3}\left(\frac{mc}{h}\right)^3 mc^2 \left(\frac{3}{8\pi}\right)^{5/3}\left(\frac{h}{mc}\right)^5 \frac{8}{5} n^{5/3}$$

Here, 'm' is the mass of the particle that is degenerate

This simplifies to

$$P \sim \frac{h^2}{m} n^{5/3}$$

For neutron/protons $m_0 \sim 2000\, m_e$, so $P_n \ll P_e$ — degeneracy is not important for protons and neutrons until we get to really high densities

A bit more...

Useful discriminant: degeneracy parameter

$$\eta = \frac{\mu - m_e c^2}{kT}$$

$\eta > 1$ is degenerate

Coulomb effects

- compare coulomb potential between two ions w/ kT

$$\Gamma_c \equiv \frac{Z^2 e^2}{\underbrace{a}_{\text{separation}} kT}$$

then define a via $\frac{4\pi}{3} a^3 \sim \frac{1}{\underbrace{n_I}_{\text{ion \# density}}}$ (Wigner-Seitz sphere)

$\Gamma_c > 1$ means Coulomb effects become important

$\Gamma_c \gg 1$ crystalization sets in (important for WD cooling)

---

radiation important when $\frac{1}{3} a T^4 \sim \frac{\rho k T}{\mu m_0}$

---

pressure ionization: in dense gas, electric field of one atom disrupts neighboring atoms — ionization sets in -

set $a \sim$ Bohr radius, then $\rho \sim 1\ g\ cm^{-3}$ for ionization

---

+ interactive w/ helmeos

We ignored ionization so far...

chemical equilibrium

$$\sum \mu_i d\tilde{N}_i = 0$$ — this takes into account changes in # due to reactions

consider

equilibrium

$$H^+ + e^+ \longleftrightarrow H^0 + \gamma$$

we can write this as

$$1H^+ + 1e^- - 1H^0 = 0$$ (ignoring photon)

or $\sum_i \nu_i C_i = 0$

stoichiometric coefficents

the concentrations N_i are constrained in the same way

$$\frac{dN_i}{\nu_i} = \frac{dN_1}{\nu_1}$$

some component i responds to a change in '1'

then

$$\sum \mu_i dN_i = \sum \mu_i \frac{dN_1}{\nu_i} \nu_i = \frac{dN_1}{\nu_1} \sum_i \mu_i \nu_i = 0$$

so $\sum_i \mu_i \nu_i = 0$ in chemical equilibrium

We'll take the reference energy as the $H^+ + e^-$ state having $\varepsilon_0 = 0$

Then for H^0 it is $-\chi_H = -13.6\,eV$

ionization potential of H

Now for degeneracy (of state) parameters,

H^+: $g_+ = 1$ by choice (fixed reference)

e^-: $g_- = 2$ the electron can be in the same direction or opposite the proton } we can swap those w/o effect

H^0: $g_0 = 2$ electron + proton spins aligned or anti-aligned (ignore 21 cm energy)

chemical equilibrium:

$$\mu_- + \mu_+ - \mu_0 = 0$$

recall for an ideal gas

$$n = \frac{\cancel{4}\pi g(2mkT)^{3/2}}{h^3}\frac{\sqrt{\pi}}{\cancel{4}}e^{\mu/kT}e^{-\varepsilon_0/kT}$$

$$= \frac{g(2\pi mkT)^{3/2}}{h^3}e^{\mu/kT}e^{-\varepsilon_0/kT}$$

Then

$$n_+ = \frac{(2\pi m_p kT)^{3/2}}{h^3} e^{\mu_+/kT}$$

$$n_- = \frac{2\cdot(2\pi m_e kT)^{3/2}}{h^3} e^{\mu_-/kT}$$

$$n_0 = \frac{2\cdot(2\pi m_H kT)^{3/2}}{h^3} e^{\mu_0/kT} e^{\chi_H/kT}$$

$m_H = m_p + m_e$

now consider

$$\frac{n_+ n_-}{n_0} = \frac{(2\pi kT)^{3/2}}{h^3}\Bigg(\underbrace{\frac{m_e m_p}{m_p + m_e}}_{\sim m_e}\Bigg)^{3/2} e^{\overbrace{(\mu_+ + \mu_- - \mu_0)}^{=0!\ \text{by chemical equilibrium}}/kT} e^{-\chi_H/kT}$$

then we got the Saha equation

$$\frac{n_+ n_-}{n_0} = \frac{(2\pi kT)^{3/2}}{h^3} m_e^{3/2} e^{-\chi_H/kT}$$

now charge neutrality means $n_+ = n_-$

number of nucleons must be conserved, $n = n_0 + n_+$

divide by n, then

$$1 = \frac{n_0}{n} + y$$

define

$$y \equiv \frac{n_+}{n_0 + n_+} = \frac{n_+}{n} = \frac{n_-}{n}$$

then

$$\frac{n_+ n_-}{n_0} = \frac{(yn)(yn)}{n(1-y)} = \frac{y^2 n}{1-y} = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_H/kT}$$

we can solve this for ionization fraction, y, given n