Convection... but first, some thermodynamics

We will need some derivatives for convection

specific heat

$$c_\alpha = \left(\frac{dq}{dT}\right)_\alpha \qquad \text{where} \quad dq = de + P\,d\left(\frac{1}{\rho}\right)$$

dq — units of erg/g

$$c_v = \frac{dq}{dT}\bigg|_\rho = \frac{\partial e}{\partial T}\bigg|_\rho$$

v — specific volume

introducing specific enthalpy, $h = e + \frac{P}{\rho}$

$$dh = de + d\left(\frac{P}{\rho}\right) = \underline{de} + P\,d\left(\frac{1}{\rho}\right) + \frac{1}{\rho}\,dP$$

$$\therefore \quad dq = \left(dh - \frac{1}{\rho}\,dP\right)$$

and $c_p = \frac{dq}{dT}\Big|_P = \frac{\partial h}{\partial T}\Big|_P$

one can show that $c_p - c_v = \frac{P}{\rho T}\frac{\chi_T^2}{\chi_\rho}$ (note $\chi_T = \chi_\rho = 1$ for ideal gas $\therefore c_p - c_v = \frac{k}{\mu m}$)

w/ $\chi_T \equiv \frac{\partial \ln P}{\partial \ln T}\Big|_\rho \qquad \chi_\rho \equiv \frac{\partial \ln P}{\partial \ln \rho}\Big|_T$

ratio of specific heats: $\gamma = \frac{c_p}{c_v}$ — not necessarily constant

2.

Adiabatic exponents

$$\Gamma_1 \equiv \left.\frac{\partial \log P}{\partial \log \rho}\right|_s$$

$$\frac{\Gamma_2}{\Gamma_2 - 1} \equiv \left.\frac{\partial \log P}{\partial \log T}\right|_s \equiv \frac{1}{\nabla_{ad}}$$

$$\Gamma_3 - 1 \equiv \left.\frac{\partial \log T}{\partial \log \rho}\right|_s$$

$$\nabla_{ad} = \frac{\Gamma_2 - 1}{\Gamma_2} = \frac{\Gamma_3 - 1}{\Gamma_1}$$

many more relations exist

Note: for an ideal gas, $\gamma = \Gamma_1 = \Gamma_2 = \Gamma_3$
but this is not true in general

If we are adiabatic, then we can write out equation of state as

$$P = K\rho^{\Gamma_1}$$

$\uparrow$ constant that depends on entropy

What about the sound speed?

we'll look at linear acoustics

The Euler equations appear as

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0 \qquad \text{(conservation of mass)}$$

$$\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u u)}{\partial x} + \frac{\partial P}{\partial x} = 0 \qquad \text{(conservation of momentum)}$$

$$\frac{\partial (\rho E)}{\partial t} + \frac{\partial (\rho u E + u P)}{\partial x} = 0 \qquad \text{(conservation of energy)}$$

if we assume isentropic flow, then we can replace the energy equation by

$$P = K \rho^{\Gamma_1}$$

expanding the first two, we have

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial P}{\partial x} = 0$$

consider a stationary background w/ small perturbations

$$\rho = \rho_0 + \delta\rho$$

$$P = P_0 + \delta p$$

$$u = 0 + \delta u$$

To first order

(a) $$\frac{\partial \delta\rho}{\partial t} + \rho_0 \frac{\partial \delta v}{\partial x} = 0$$

(b) $$\frac{\partial \delta v}{\partial t} + \frac{1}{\rho_0}\frac{\partial \delta P}{\partial x} = 0$$

eliminating δv by differentiating (a) wrt t and (b) wrt x

$$\left.\begin{aligned} \frac{\partial^2 \delta\rho}{\partial t^2} + \rho_0 \frac{\partial^2 \delta v}{\partial x \partial t} &= 0 \\ \frac{\partial^2 \delta v}{\partial x \partial t} + \frac{1}{\rho_0}\frac{\partial^2 \delta P}{\partial x^2} &= 0 \end{aligned}\right\} \quad \frac{\partial^2 \delta\rho}{\partial t^2} = \frac{\partial^2 \delta P}{\partial x^2}$$

since we are isentropic, $P(\rho, s) = P(\rho) \longrightarrow P = K\rho^{\Gamma_1}$

$$\delta P = \Gamma_1 K \rho^{\Gamma_1 - 1}\,\delta\rho = \frac{\Gamma_1 P}{\rho}\,\delta\rho$$

$$\therefore \quad \frac{\partial^2 \delta P}{\partial t^2} = \frac{\Gamma_1 P}{\rho}\frac{\partial^2 \delta\rho}{\partial x^2}$$

this is a wave equation. We define $c_s^2 \equiv \frac{\Gamma_1 P}{\rho}$ as the speed of sound

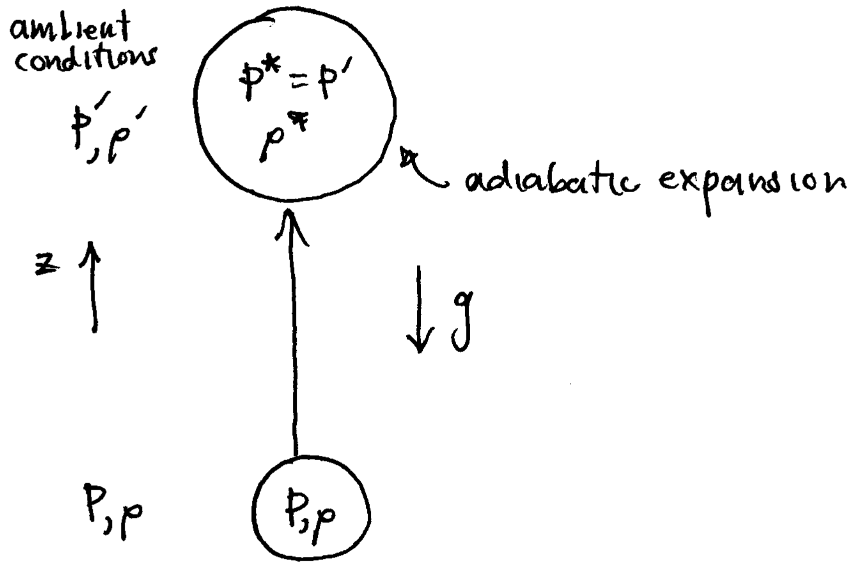
$$\therefore \quad c_s = \sqrt{\frac{\Gamma_1 P}{\rho}}$$

this admits propagating waves as solution (linear acoustic waves)

Real convection is messy. We'll look at the idealized case and look for a condition for a fluid parcel to be convectively unstable.

This is different than HKT's approach (we follow Choudhuri and C&O)

Q: If we displace a fluid parcel upwards, will it continue to rise?



If the motion is adiabatic, then the fluid parcel always remains in pressure equilibrium w/ its surroundings

Adiabatic expansion implies $P = K\rho^{\Gamma_1}$

$\therefore$ after rising, the new density is

$$\rho^* = \rho\left(\frac{P'}{P}\right)^{1/\Gamma_1}$$

writing $P' = P + \frac{dP}{dr}\Delta r$, we have

$$\rho^* = \rho\left[1 + \frac{\Delta r}{P}\frac{dP}{dr}\right]^{1/\Gamma_1} \sim \rho + \frac{\rho}{\Gamma_1 P}\frac{dP}{dr}\Delta r \qquad (\text{for } \Delta r \ll H)$$

scale height

6.

How does the ambient density vary?

$$\rho' = \rho + \frac{d\rho}{dr}\,\Delta r$$

To make further progress we'll assume an ideal gas (we'll relax this later)

$$\rho = \frac{\mu m P}{kT} \qquad \left(\rho = \rho(T, P) \right)$$

We also take μ as constant — much more complicated behavior results if μ varies

$$\frac{d\rho}{dr} = \left.\frac{\partial \rho}{\partial P}\right|_T \frac{dP}{dr} + \left.\frac{\partial \rho}{\partial T}\right|_P \frac{dT}{dr}$$

$$= \frac{\mu m}{kT}\frac{dP}{dr} - \frac{\mu m P}{kT^2}\frac{dT}{dr}$$

$$= \frac{\rho}{P}\frac{dP}{dr} - \frac{\rho}{T}\frac{dT}{dr}$$

$$\therefore \quad \rho' = \rho + \left\{ \frac{\rho}{P}\frac{dP}{dr} - \frac{\rho}{T}\frac{dT}{dr} \right\} \Delta r$$

Our parcel will continue to rise as long as it is booyant, at its new height

convective $\underline{\underline{\text{instability}}}$: $\rho^* - \rho' < 0$

$$\rho^* - \rho' = \frac{\rho}{\gamma P}\frac{dP}{dr}\,\Delta r - \left\{ \frac{\rho}{P}\frac{dP}{dr} - \frac{\rho}{T}\frac{dT}{dr} \right\} \Delta r$$

$$= \left[\left(\frac{1}{\gamma} - 1 \right) \frac{\rho}{P}\frac{dP}{dr} + \frac{\rho}{T}\frac{dT}{dr} \right] \Delta r$$

note: $\Delta r > 0$ since we displaced upward

$$\therefore \quad \left(\frac{1}{\gamma} - 1 \right) \frac{\rho}{P}\frac{dP}{dr} < - \frac{\rho}{T}\frac{dT}{dr}$$

Note that both $\frac{dP}{dr}$ and $\frac{dT}{dr} < 0$.

$\therefore$ we can write this as

$$\left|\frac{dT}{dr}\right| > \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\left|\frac{dP}{dr}\right| \qquad \left(\text{using } \left|\frac{dT}{dr}\right| = -\frac{dT}{dr}, \ldots\right)$$

Essentially, this is saying that if the temperature profile in the star is steep, then we are unstable to convection.

How does this compare to an adiabatic $\frac{dT}{dr}$?

start w/ $\frac{dP}{dr} = \frac{P}{\rho}\frac{d\rho}{dr} + \frac{P}{T}\frac{dT}{dr}$ (remember, we are still assuming an ideal gas)

but also, if we are adiabatic, then

$P = K\rho^{\Gamma_1}$, so (using 8)

$$\frac{dP}{dr} = \gamma\frac{P}{\rho}\frac{d\rho}{dr} \longrightarrow \frac{d\rho}{dr} = \frac{\rho}{\gamma P}\frac{dP}{dr}$$

$$\therefore \quad \frac{dP}{dr} = \frac{1}{\gamma}\frac{dP}{dr} + \frac{P}{T}\frac{dT}{dr}$$

or

$$\left.\frac{dT}{dr}\right|_{ad} = \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}$$

we use the 'ad' subscript to indicate this was derived under adiabatic conditions

so we have

$$\left|\frac{dT}{dr}\right| > \left|\frac{dT}{dr}\right|_{ad} \quad \text{for convection to take place}$$

What does this mean?

The adiabatic temperature gradient is the temperature profile we realize if the star has constant entropy

$$\left.\frac{dT}{dr}\right|_{ad} = \left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}$$

we are comparing the actual temperature gradient in the star, dT/dr, to the adiabatic gradient.

if $\quad \left|\frac{dT}{dr}\right| > \left|\frac{dT}{dr}\right|_{ad} \quad$ or $\quad \frac{dT}{dr} < \left.\frac{dT}{dr}\right|_{ad}$

then convection takes place (we are convectively unstable)

This means that very steep temperature gradients lead to convection.

Often, <u>if</u> we are convecting, then it will dominate, and radiation will not be significant (except near the surface)

Convection is also very efficient, so if $\frac{dT}{dr} < \left.\frac{dT}{dr}\right|_{ad}$, then convection can typically carry the entire stellar luminosity L.

9. What about a real gas?

We had $\rho^* = \rho + \frac{\rho}{\Gamma_1 P} \frac{dP}{dr} \Delta r$

and $\rho' = \rho + \frac{d\rho}{dr} \Delta r$

Now, $\frac{dP}{dr} = \left.\frac{\partial P}{\partial T}\right|_\rho \frac{dT}{dr} + \left.\frac{\partial P}{\partial \rho}\right|_T \frac{d\rho}{dr}$

$= \frac{P}{T} \chi_T \frac{dT}{dr} + \frac{P}{\rho} \chi_\rho \frac{d\rho}{dr}$

[note for χ, we can think of this as $P = P_0 \rho^{\chi_\rho} T^{\chi_T}$]

$\therefore \frac{d\rho}{dr} = \frac{1}{\chi_\rho} \frac{\rho}{P} \left(\frac{dP}{dr} - \frac{P}{T} \chi_T \frac{dT}{dr} \right)$

so $\rho' = \rho + \left\{ \frac{\rho}{\chi_\rho P} \frac{dP}{dr} - \frac{\rho}{T} \frac{\chi_T}{\chi_\rho} \frac{dT}{dr} \right\} \Delta r$

and convection again results if

$\rho^* - \rho' < 0$

$\rho^* - \rho' = \left\{ \frac{\rho}{\Gamma_1 P} \frac{dP}{dr} - \frac{\rho}{\chi_\rho P} \frac{dP}{dr} + \frac{\rho}{T} \frac{\chi_T}{\chi_\rho} \frac{dT}{dr} \right\} \Delta r$

so we require

$\frac{\rho}{P} \left[\frac{1}{\Gamma_1} - \frac{1}{\chi_\rho} \right] \frac{dP}{dr} < -\frac{\rho}{T} \frac{\chi_T}{\chi_\rho} \frac{dT}{dr}$

or $\underbrace{\left| \frac{dT}{dr} \right|}_{\text{actual}} > \frac{T}{P} \left(\frac{1}{\chi_\rho} - \frac{1}{\Gamma_1} \right) \frac{\chi_\rho}{\chi_T} \left| \frac{dP}{dr} \right|$

What is

$$\left(\frac{1}{\chi_\rho} - \frac{1}{\Gamma_1}\right)\frac{\chi_\rho}{\chi_T} \, ?$$

$$= \left(1 - \frac{\chi_\rho}{\Gamma_1}\right)\frac{1}{\chi_T}$$

but HKT Eq 3.98 says $\Gamma_1 = \chi_T(\Gamma_3 - 1) + \chi_\rho$

$$\therefore \quad 1 - \frac{\chi_\rho}{\Gamma_1} = \chi_T \frac{\Gamma_3 - 1}{\Gamma_1} = \chi_T \nabla_{ad}$$

so we have

$$\left|\frac{dT}{dr}\right| > \frac{T}{P}\nabla_{ad}\left|\frac{dP}{dr}\right| = \frac{T}{P}\left(1 - \frac{1}{\Gamma_2}\right)\left(\frac{dP}{dr}\right)$$

or

$$\frac{dT}{dr} < \frac{T}{P}\nabla_{ad}\frac{dP}{dr}$$

but since $\nabla_{ad} \equiv \left.\frac{\partial \ln T}{\partial \ln P}\right|_s$, we have

$$\frac{T}{P}\left.\frac{\partial \ln T}{\partial \ln P}\right|_s \frac{dP}{dr} = \left.\frac{\partial T}{\partial P}\right|_s \frac{dP}{dr} = \left.\frac{dT}{dr}\right|_{ad}$$

so, again we find that

$$\frac{dT}{dr} < \left.\frac{dT}{dr}\right|_{ad} \quad \text{for convection}$$

$$\left(\left|\frac{dT}{dr}\right| > \left|\frac{dT}{dr}\right|_{ad} \right.$$ — steeper than adiabatic gradient needed

Playing around some more, we have

$$\frac{dT}{dr} < \frac{T}{P} \nabla_{ad} \frac{dP}{dr} \qquad \left(\text{since } \frac{dT}{dr} < 0 \text{ and } \frac{dP}{dr} < 0\right)$$

or

$$\frac{P}{T}\frac{dT}{dP} > \nabla_{ad} \qquad \text{(inequality switches direction because we divided by } dP/dr < 0\text{)}$$

or

$\nabla > \nabla_{ad}$ for convection!

This was all idealized!

neglected:
- overshoot
- composition gradients
- radiative leakage
- turbulence
- semi-convection

$\vdots$

Note that for an isothermal atmosphere,

$$\nabla = \frac{d\log T}{d\log P} = \frac{\left(d\log T/dr\right)}{\left(d\log P/dr\right)} = \frac{0}{\left(d\log P/dr\right)} = 0$$

$\therefore$ an isothermal atmosphere is not convective

Consider

$$L = -\frac{16\pi a c r^2}{3\kappa\rho} T^3 \frac{dT}{dr} \quad \text{(radiation transfer)}$$

Notice that:

- if κ is really large, then $\left|\frac{dT}{dr}\right|$ also must be large to carry the same L
- if L is large, then $\left|\frac{dT}{dr}\right|$ must be large to carry the energy radiatively

but dT/dr cannot grow without bound — convection kicks in — it is an instability

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We can also see (show) that convection arises when you have high entropy material beneath low entropy material.

Efficient convection will tend to make the region isentropic (s = constant)

Also note: when convection is operating, and nuclear products are created, they will be distributed throughout the convective zone.

Adiabatic T gradient for an ideal gas

$$\left.\frac{dT}{dr}\right|_{ad} = -\left(1-\frac{1}{\gamma}\right)\frac{T}{P}\left|\frac{dP}{dr}\right|$$

$$= -\left(1-\frac{1}{\gamma}\right)\frac{\mu m_0}{\rho k}\rho|g|$$

$$= -\left(1-\frac{1}{\gamma}\right)\frac{\mu m_0}{k}\frac{GM}{r^2}$$

now, recall $\gamma = \frac{c_p}{c_v}$ and $c_p - c_v = \frac{k}{\mu m_0}$ (both for ideal gas)

then

$$\left.\frac{dT}{dr}\right|_{ad} = -\left(1-\frac{c_v}{c_p}\right)\frac{1}{c_p-c_v}|g| = -\frac{|g|}{c_p}$$

regions where ionization is occuring have a large c_p

— ionization means that dT does not rise as fast as de, since some of e goes into ionization

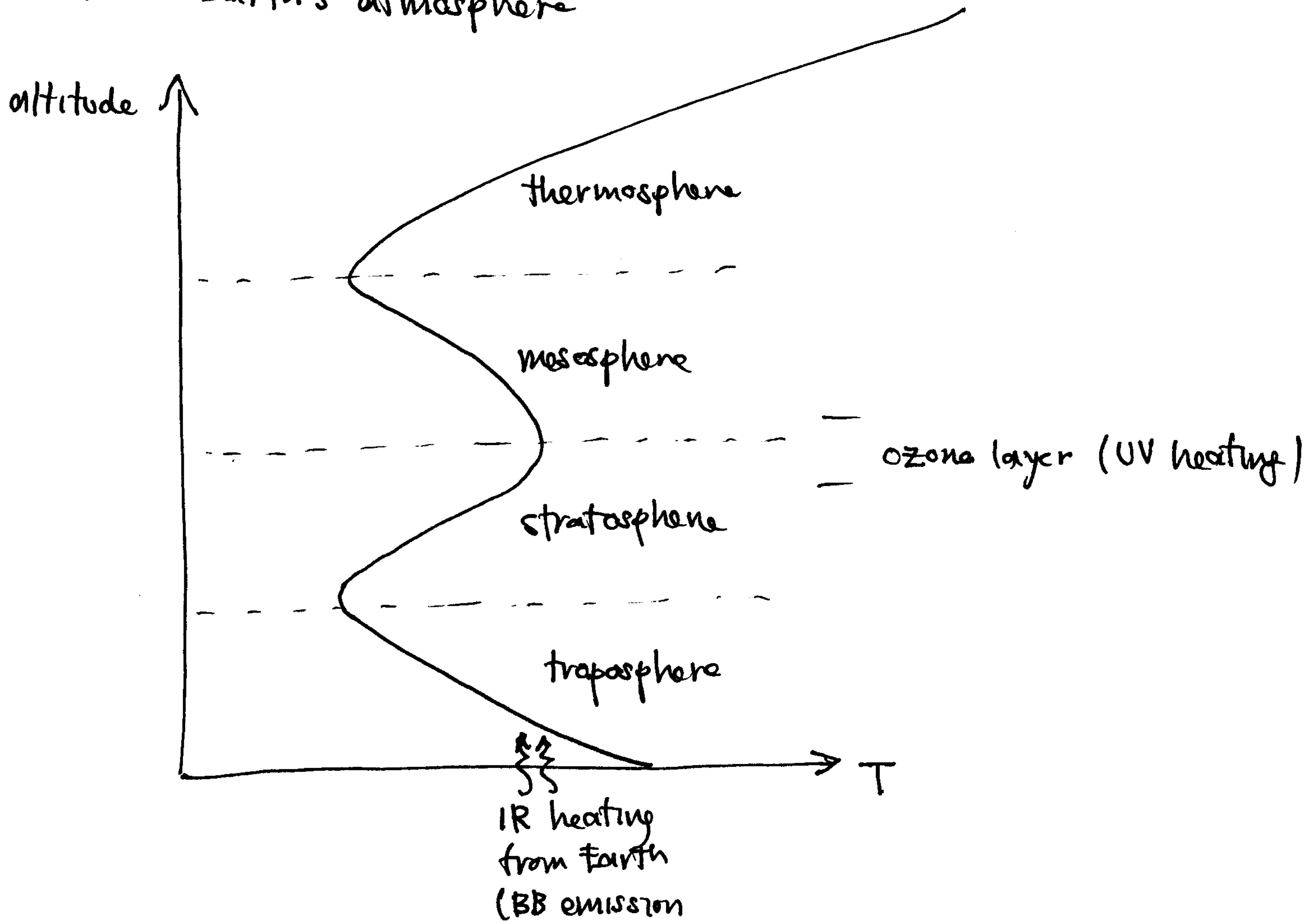
$\therefore c_x \sim \frac{\partial e}{\partial T}$ increases

then ionization means large c_p and $\therefore$ small $\left|\frac{dT}{dr}\right|_{ad}$

— convection becomes easier

Finally, dT/dr can be large for energy generation rates that are very T sensitive — e.g. CNO or $3-\alpha$

Aside: Earth's atmosphere



troposphere: $\frac{dT}{dr} < 0$ and $\underbrace{\frac{dP}{dr} < 0}_{\text{HSE}}$ and is convectively unstable

stratosphere: $\frac{dT}{dr} > 0$ and $\frac{dP}{dr} < 0$ making

$$\nabla \equiv \frac{d\log T}{d\log P} < 0 \text{ — convectively stable}$$

We can have weather in the troposphere because of convection (allows cloud formation)

What now?

We've only shown that an instability exists.

It's easy to see that the fluid element can transport heat.

Conservation of mass requires both upward (hot) and downward (cooler) bubbles — a complex overturning motion.

To incorporate these ideas into our stellar models, we need a way of finding the dT/dr necessary to carry the luminosity — this will replace or augment the radiation equation

$\therefore$ we need to find the convective flux mixing length theory (MLT)

– this is heuristic model for describing convection in stellar evolution codes.

Here MLT is a local process that can provide an estimate for the convective flux

Note: real life is 3-d – convection is characterized by overturning fluid motions carrying the energy.

We'll consider a simple idealized description of MLT to get a flavor (following C&O)

convection results if $\rho^* < \rho'$ (bubble remains buoyant)

$$\left|\frac{dT}{dr}\right| - \left|\frac{dT}{dr}\right|_{ad} > 0$$

↑ actual

$$\frac{dT}{dr} - \left.\frac{dT}{dr}\right|_{ad} < 0$$

define $\delta T \equiv \left(\left.\frac{dT}{dr}\right|_{ad} - \frac{dT}{dr} \right) dr = \delta\left(\frac{dT}{dr}\right) dr$ (some texts write $\Delta\nabla$)

$\delta T > 0$ means superadiabatic $\longrightarrow$ convection

Assume that our bubble rises a distance

$$\ell = \underbrace{\alpha H}_{}$$

H — scale height

at which point it dissipates and gives its excess heat to the surrounding fluid

ℓ is the mixing length

α is a parameter, assumed $\alpha \sim 1$

The exchange of heat to the surroundings is at constant pressure, so

$$\delta Q = (c_p \delta T)\rho$$

δQ — heat release / volume

taking $dr \sim \ell$ and assuming a velocity of the convective bubble $\bar{v}_c$, the heat flux (convective flux) is

$$F_c = \delta Q\, \bar{v}_c \qquad \text{(energy / time / area)}$$

$$= c_p \delta T \underbrace{\rho\, \bar{v}_c}_{\text{mass flux}}$$

We estimate $\bar{v}_c$ from the forces on the bubble

$$\delta P = \frac{P}{\rho}\delta\rho + \frac{P}{T}\delta T \qquad (\text{ignoring } \mu)$$

but $\delta P = P_{bubble} - P_{ambient} = 0$

$$\therefore \delta\rho = -\frac{\rho}{T}\delta T$$

The buoyant force is

$$f_{buoy} = -g\underset{\text{difference wrt background}}{\delta\rho}$$

$$= +\frac{\rho g}{T}\delta T$$

since the bubble starts out nearly in equilibrium wrt the surroundings, $f_{buoy} \sim 0$ initially

Averaging over initial and final,

$$\langle f_{buoy} \rangle \sim \frac{1}{2}\frac{\rho g}{T}\delta T_f$$

The work goes into kinetic energy:

$$\frac{1}{2}\rho v_f^2 = \langle f_{buoy} \rangle \ell$$

$$\therefore \quad \bar{v}_c \sim \left(\frac{2\beta\langle f_{buoy}\rangle \ell}{\rho}\right)^{1/2}$$

β is some parameter representing how v^2 changes over a mixing length: $0 < \beta < 1$

taking $dr = \ell$

$$\bar{v}_c \sim \left(\beta\frac{g}{T}\delta T\ell\right)^{1/2} \sim \left(\beta\frac{g}{T}\delta\left(\frac{dT}{dr}\right)\right)^{1/2}\underset{\propto H}{\ell}$$

The flux is then

$$F_c = c_p \,\delta T \rho v_c$$

$$= c_p \,\delta T \rho \left[\beta \frac{g}{T} \delta\left(\frac{dT}{dr}\right)\right]^{1/2} l$$

$$= c_p\left[\delta\left(\frac{dT}{dr}\right) l\right] \rho \left[\beta \frac{g}{T} \delta\left(\frac{dT}{dr}\right)\right]^{1/2} l$$

$$= c_p \rho \left(\frac{\beta g}{T}\right)^{1/2} \left[\delta\left(\frac{dT}{dr}\right)\right]^{3/2} \alpha^2 H^2$$

recall
$H \sim \frac{P}{\rho g} = \frac{kT}{\mu m_u g}$

$$\sim \rho c_p \beta^{1/2} \left(\frac{T}{g}\right)^{3/2} \left(\frac{k}{\mu m_u}\right)^2 \alpha^2 \left[\delta\left(\frac{dT}{dr}\right)\right]^{3/2}$$

Note the strong dependence on $\delta\left(\frac{dT}{dr}\right)$ and α

Let's estimate $\delta\left(\frac{dT}{dr}\right)$ in the Sun

Imagine all flux is carried by convection

$$F_c = \frac{L(r)}{4\pi r^2} = \rho c_p \beta^{1/2} \left(\frac{T}{g}\right)^{3/2} \left(\frac{k}{\mu m_u}\right)^2 \alpha^2 \left[\delta\left(\frac{dT}{dr}\right)\right]^{3/2}$$

$$\therefore\ \delta\left(\frac{dT}{dr}\right) = \left[\frac{L}{4\pi r^2}\left(\frac{\mu m_u}{k}\right)^2 \left(\frac{g}{T}\right)^{3/2} \beta^{-1/2} \frac{1}{\rho c_p \alpha^2}\right]^{2/3}$$

To get a feel for how superadiabatic we need to be, recall from before

$$\left.\frac{dT}{dr}\right|_{ad} = -\frac{g}{c_p}$$

$$\therefore\ \frac{\delta\left(\frac{dT}{dr}\right)}{\left|\frac{dT}{dr}\right|_{ad}} = \left(\frac{L}{4\pi r^2}\right)^{2/3} \left(\frac{\mu m_u}{k}\right)^{4/3} \frac{1}{T} \beta^{-1/3} \frac{1}{(\rho\alpha^2)^{2/3}} c_p^{1/3}$$

Take $\alpha \sim \beta \sim 1$

the convective zone is in the outer part of the Sun — most of the mass is enclosed,

$\therefore$ $M \sim 1 M_\odot$

$L \sim 1 L_\odot$ — all energy generated in core

$r \sim 0.75 R_\odot$

$c_p = \frac{5}{2}\frac{k}{\mu m_u}$

$p \sim 3\times10^{13}\ \text{dyn/cm}^2$

$\rho \sim 0.1\ \text{g/cm}^3$

$\mu \sim 0.6$

$T \sim 1.8\times10^6\ \text{K}$

Putting in these numbers

$$\left.\frac{dT}{dr}\right|_{ad} \sim 1.4\times10^{-4}\ K\ cm^{-1}$$

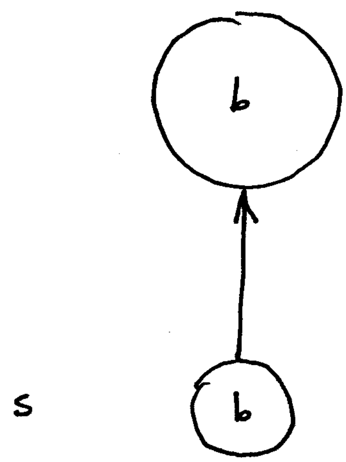
$$\delta\left(\frac{dT}{dr}\right) \sim 8\times10^{-11}\ K\ cm^{-1}$$

$\therefore$ the amount by which the T gradient needs to be superadiabatic is

$$\frac{\delta\left(dT/dr\right)}{\left|dT/dr\right|_{ad}} \sim 6\times10^{-7}$$

$\therefore$ in stellar interiors taking the T gradient to be adiabatic is usually a good approx

What about composition? (following Kippenhahn & Weigert)



consider (again) the change in density of the bubble

$$D\rho = \left\{ \underbrace{\left(\frac{d\rho}{dr}\right)_b}_{\text{bubble}} - \underbrace{\left(\frac{d\rho}{dr}\right)_s}_{\text{surroundings}} \right\} \Delta r \underbrace{< 0}_{\substack{\text{to be}\\ \text{buoyant}\\ \text{(convective)}}}$$

consider $\rho = \rho(P, T, \mu)$ ← mean molecular weight

$$\frac{d\rho}{\rho} = \underbrace{\left(\frac{\partial \ln \rho}{\partial \ln P}\right)\Bigg|_{T,\mu}}_{\equiv \alpha} \frac{dP}{P} + \underbrace{\left(\frac{\partial \ln \rho}{\partial \ln T}\right)\Bigg|_{P,\mu}}_{\equiv -\delta} \frac{dT}{T} + \underbrace{\left(\frac{\partial \ln \rho}{\partial \ln \mu}\right)\Bigg|_{P,T}}_{\equiv \phi} \frac{d\mu}{\mu}$$

for an ideal gas, $\alpha = \delta = \phi = 1$

μ doesn't change inside the bubble (no reactions) but μ is stratified in the background

Then our condition is

$$\left(\frac{d\rho}{dr}\right)_b - \left(\frac{d\rho}{dr}\right)_s =$$

$$\left(\frac{\alpha}{P}\frac{dP}{dr}\right)_b - \left(\frac{\delta}{T}\frac{dT}{dr}\right)_b - \left(\frac{\alpha}{P}\frac{dP}{dr}\right)_s + \left(\frac{\delta}{T}\frac{dT}{dr}\right)_s - \left(\frac{\phi}{\mu}\frac{d\mu}{dr}\right)_s$$

We always remain in pressure equilibrium, so $\frac{dP}{dr}$ is the same inside & outside the bubble

$\therefore$ we have

instability

$$-\left(\frac{\delta}{T}\frac{dT}{dr}\right)_b + \left(\frac{\delta}{T}\frac{dT}{dr}\right)_s - \left(\frac{\phi}{\mu}\frac{d\mu}{dr}\right)_s < 0$$

multiply be the scale height $-P\frac{dr}{dP}$ (>0)

$$\delta\left(\frac{d\ln T}{d\ln P}\right)_b - \delta\left(\frac{d\ln T}{d\ln P}\right)_s + \phi\left(\frac{d\ln\mu}{d\ln P}\right)_s < 0$$

assuming $\delta > 0$, we divide by δ

$$\left(\frac{d\ln T}{d\ln P}\right)_s > \left(\frac{d\ln T}{d\ln P}\right)_b + \frac{\phi}{\delta}\left(\frac{d\ln\mu}{d\ln P}\right)_s \quad \text{for instability}$$

this is

$$\nabla > \nabla_{ad} + \frac{\phi}{\delta}\nabla_\mu \quad \text{w/} \quad \nabla_\mu \equiv \left(\frac{d\ln\mu}{d\ln P}\right)_{\text{surroundings}}$$

actual log slope of μ w/ P in the star

The effect of a composition gradient is as follows:

- since μ usually increases inwards (heavy elements are produced in the center of stars),

  μ increase w/ P, $\therefore \nabla_\mu > 0$

- Our convection criterion now says it is harder to convect:

$$\nabla > \nabla_{ad} + \underbrace{\frac{\phi}{\delta} \nabla_\mu}_{>0}$$

$\therefore$ the normal composition gradient has a stabilizing effect against convection!

fun things can happen when our previous condition

$$\nabla > \nabla_{ad} \quad \text{(Schwarzschild)}$$

says convect but this new one

$$\nabla > \nabla_{ad} + \frac{\phi}{\delta} \nabla_\mu \quad \text{(Ledoux)}$$

says stable — this gives rise to <u>semi-convection</u>
(also called <u>double-diffusive convection</u> in planetary / Earth science) — see Mirouh et al 2011 ApJ

(sometimes μ can increase w/ radius, leading to <u>"fingering convection"</u> or <u>thermohaline convection</u> — diffusion can trigger an instability even when Ledoux says 'no'
— see Brown, Garaud, and Stellmach (e.g.))

Our equations for stellar structure are now

$$\frac{dr}{dM} = \frac{1}{4\pi r^2 \rho}$$

$$\frac{dP}{dM} = -\frac{GM}{4\pi r^4}$$

$$\frac{dL}{dM} = \epsilon$$

$$\frac{dT}{dM} = -\frac{3}{4ac}\frac{\bar{\kappa}}{T^3}\frac{L(r)}{(4\pi r^2)^2} \quad \text{(radiation)}$$

— or —

$$\frac{dT}{dM} = \frac{T}{P}\nabla_{ad}\frac{dP}{dM} \quad \text{(convection)}$$

we'll use this when $\nabla > \nabla_{ad}$
(in fact this just says $\nabla = \nabla_{ad}$)

We could instead use the actual expression from MLT here

Note: these are time-independent
real stars have time dependent
- if evolution is slow, we can make snapshots, change composition, new snapshot, ...

Later we'll look at some of the time-dependent terms