PHY 521: Stars

Homework #7

Due: 2025-11-20 (by the end of the scheduled class period)

Be sure to show all work. Assignment is turned in via Brightspace assignment upload. You should make a single, legible PDF with your solutions. You must show your work and include units with all numbers, all throughout each problem.

While you are free to discuss problems with your classmates, your work must be your own. Any use of AI/LLM must be disclosed. All calculations (including any code you may write) and writing up of the problem solution must be your own work.

Unless otherwise noted, all problems carry equal weight.

1. *Electron heat capacity.* (10 pts.) When we computed the WD cooling timescale, we ignored the heat capacity of the electrons. Here we can check how good to this approximation is. In homework 3, we computed the temperature-correction to the pressure in the non-relativistic limit and found:

$$P_e = \frac{h^2}{20m}\left(\frac{3}{\pi}\right)^{2/3} n_e^{5/3}\left[1 + \frac{40\pi^2}{(3/\pi)^{4/3}}\frac{m^2}{h^4}\frac{(kT)^2}{n_e^{4/3}}\right] \tag{1}$$

We also know that the energy and pressure are related as: $\rho e_e = (3/2)P_e$.

Using this result, compute the specific heat, $c_v = \partial e/\partial T|_\rho$ and evaluate it for the conditions appropriate in a white dwarf. Compare to the result we found for the ions and confirm that our approximation was valid.

2. At high-temperatures, the energy release from the hot-CNO cycle is

$$\epsilon_H = 5.9 \times 10^{15} Z_{\mathrm{CNO}} \text{ erg g}^{-1}\text{ s}^{-1}$$

Here we will approximate this result.

The hot-CNO cycle becomes β-limited, and the decay of two isotopes, ^{14}O and ^{15}O, set the timescale. These have half-lives of 71 s and 122 s respectively. Because these decays dominate the rates, the energy release does not depend on temperature, and instead only on the mass fraction of CNO nuclei present, Z_{CNO}.

In the parts below, X represents the mass fraction of a species and $Y = X/A$ represents the molar fraction of a species (with A the atomic weight of that species).

(a) We can take every nucleus in the cycle to be in a steady-state, and the only two that have any significant abundance are ^{14}O and ^{15}O (because these decays are the waiting points). Take $Z_{\mathrm{CNO}} = X(^{14}\mathrm{O}) + X(^{15}\mathrm{O})$. Using $dY(^{14}\mathrm{O})/dt = dY(^{15}\mathrm{O})/dt = 0$, show that the *mass fraction* of ^{14}O is related to Z_{CNO} as

$$X(^{14}\mathrm{O}) \sim Z_{\mathrm{CNO}}/2.84 \tag{2}$$

You may want to review the slides from class when we discussed weak rates to see how the half-life of a species is related to the decay constant used in the rate equations.

(b) If we are in equilibrium, then every time a ^{14}O nucleus decays a ^{15}O nucleus decays (this is the rate equilibrium we used above). Thus we can write the rate of destruction of ^{1}H as:

$$\frac{dY(^{1}\mathrm{H})}{dt} = -4\frac{dY(^{14}\mathrm{O})}{dt} \tag{3}$$

(or equivalently, using ^{15}O instead).

By definition, the binding energy of the ^{1}H nucleus is 0, the binding energy of the ^{4}He nucleus is 28.3 MeV, and the reaction of $4^{1}\mathrm{H} \rightarrow {}^{4}\mathrm{He}$ losses about 2.03 MeV via neutrinos. Using this information, show that ϵ_H is approximately that given above.