PHY 521: Stars

Homework #6

Due: 2025-11-06 (by the end of the scheduled class period)

Be sure to show all work. Assignment is turned in via Brightspace assignment upload. You should make a single, legible PDF with your solutions. You must show your work and include units with all numbers, all throughout each problem.

While you are free to discuss problems with your classmates, your work must be your own. Any use of AI/LLM must be disclosed. All calculations (including any code you may write) and writing up of the problem solution must be your own work.

Unless otherwise noted, all problems carry equal weight.

1. *Lane-Emden asymptotics.* We want to explore the behavior of the solution to the Lane-Emden equation, $\theta(\xi)$, near the origin. Consider an expansion of the form:

$$\theta(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + a_4\xi^4 + \dots \tag{1}$$

Our boundary conditions tell us that:

$$\theta(0) = 1 \tag{2}$$
$$\theta(\xi) = \theta(-\xi) \tag{3}$$

Use this expansion in the Lane-Emden equation to show that behavior of the solution near the origin is:

$$\theta(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 + \dots \tag{4}$$

We used this in class to avoid the singularity at $\xi = 0$ when integrating polytrope models.

Hint: For θ^n, use the binomial expansion. Group terms with the same ξ power to find the relations for the a_k in the expansion.

2. *Stellar Stability*. Here we will work through what happens when we compress a star, to see what conditions on the equation of state are necessary for stability.

   (a) Imagine compressing a star homologously, such that each radial shell compresses by the same factor. Then the new radius, r' is related to the original radius, r_0, as:

$$r' = r_0(1-\epsilon) \tag{5}$$

   where $|\epsilon| \ll 1$. The mass of a shell, $dm = 4\pi r^2 \rho dr$, doesn't change under compression. Use this fact to show the density of the compressed shell is:

$$\rho' \sim (1+3\epsilon)\rho \tag{6}$$

   (b) Hydrostatic equilibrium is easiest to work with in Lagrangian form here, since, again, the mass of a shell doesn't change. The hydrostatic pressure at the base of some shell, $m(r)$ (where $m(r)$ indicates this is the mass coordinate in the star) is:

$$P_{\rm HSE} = \int_m^{M_\star} \frac{Gmdm}{4\pi r^4} \tag{7}$$

Perturb the star as above and show that the new hydrostatic pressure, P'_{HSE} can be written in terms of the unperturbed HSE pressure, $P_{\mathrm{HSE},0}$ as:

$$P'_{\mathrm{HSE}} = (1 + \alpha\epsilon) P_{\mathrm{HSE},0} \tag{8}$$

where $\alpha > 0$ is some constant.

(**c**) If the compression is done adiabatically, then we can relate the new gas pressure, P'_g to the original, unperturbed pressure, $P_{g,0}$, in terms of the adiabatic index, Γ_1. Show that you get:

$$P'_g = (1 + \beta\Gamma_1\epsilon) P_{g,0} \tag{9}$$

where $\beta > 0$ is a constant.

(**d**) Stability requires that the gas push back against the compression, so we require that $P'_g > P'_{\mathrm{HSE}}$. What condition does this require for Γ_1 of the gas?

We've seen this result before in class, but took a very different approach to derive it then. This condition is important in regions where ionization or pair-production occurs or in the cores of massive stars.

3. *Fully convective stars.* Consider a 0.3 $M_\odot$ fully convective star. We'll treat the gas as an ideal gas, with a composition $\mu = 0.6$, and assume it behaves as a polytrope ($n = 3/2$).

(**a**) Assume a central temperature, $T_c = 10^7$ K—this should be high enough for H-fusion. By equating the ideal gas law and the polytropic equation of state (EOS) at the center of the star, find an expression for the central density in terms of the polytropic EOS constant K and T_c.

(**b**) Using the expression for mass of a polytrope in terms of K and ρ_c, find find the numerical value of K for our choice of mass and T_c.

(**c**) Using the method from class to solve the Lane-Emden equation numerically (you can just use my code at `https://zingale.github.io/stars/notebooks/polytrope/lane-emden.html`), plot the physical density and temperature in terms of physical radius (i.e., plot $\rho(r)$ and $T(r)$).

(**d**) Assuming the energy generation rate from the pp-chain, and thermal equilibrium, compute the luminosity of the star. For reference, for pp, the energy generation is:

$$\epsilon_{pp} = 2.4 \times 10^4 \, \rho X^2 T_9^{-2/3} e^{-3.38 T_9^{-1/3}} \ \mathrm{erg\ g^{-1}\ s^{1}} \tag{10}$$

This will require a numerical integration on the polytrope model produced by the ODE integration code. Any basic integration method is fine.