PHY 521: Stars

Homework #5

Due: 2025-10-28 (by the end of the scheduled class period)

Be sure to show all work. Assignment is turned in via Brightspace assignment upload. You should make a single, legible PDF with your solutions. You must show your work and include units with all numbers, all throughout each problem.

While you are free to discuss problems with your classmates, your work must be your own. Any use of AI/LLM must be disclosed. All calculations (including any code you may write) and writing up of the problem solution must be your own work.

Unless otherwise noted, all problems carry equal weight.

1. *Nuclear Reactions*

(**a**) Using classical physics arguments only, estimate the temperature required for two protons to fuse by overcoming the Coulomb barrier. Assume that once the protons are 1 fm apart, they will "see" the attractive potential well of the strong force and fuse.

(**b**) Now include the effects of quantum mechanical tunneling—imagine that the protons can fuse once the protons are within one de Broglie wavelength of one another. Expressing their momentum in terms of the de Broglie wavelength, use the energy balance as above to get and estimate for the QM temperature for fusing?

(**c**) Based on your understanding of the reactions that take place in the present-day Sun, estimate the flux of solar neutrinos that we receive at the Earth.

2. *Radiative vs. Convective Cores.* When discussing stellar evolution, we mentioned that low mass stars ($M \lesssim 1.5 M_\odot$) have radiative cores but higher mass stars have convective cores, and pointed to the fact that this is the mass where we transition from pp to CNO burning as the cause. Here we motivate this some.

Let's consider a region around the center of the star, $r \in [0, r_0]$, and if r_0 is small, we can consider the conditions here constant.

Furthermore, assume an ideal gas with a constant mean molecular weight, μ.

(**a**) Convection takes place if the temperature gradient is superadiabatic. Construct the adiabatic temperature gradient, $dT/dr|_{\mathrm{ad}}$ at the center. Use hydrostatic equilibrium to express this in terms of the central density, ρ_c and r_0.

(**b**) Expand our energy generation equation, $dL/dr = 4\pi r^2 \rho \epsilon$ about the center as

$$L = \frac{4}{3}\pi \rho_c r_0^3 \epsilon_c$$

And compute the temperature gradient needed for radiation to carry this luminosity. Assume that the opacity is a constant κ_0 (representing Thompson scattering—a good approximation in the center of a star).

(**c**) Use the condition of convective instability to find an condition of ϵ_c for convection to set in. Express this as:

$$\epsilon_c > \epsilon_{c,0} \left(\frac{T_c}{10^6\ \mathrm{K}}\right)^\alpha \left(\frac{\rho_c}{100\ \mathrm{g\ cm^{-3}}}\right)^\beta$$

where $T_6 = 10^6$ K and $\rho_2 = 10^2$ g cm^{-3} and you can assume that electron scattering dominates the opacity, $\bar{\kappa} \approx 0.2(1 + X)$ cm^2/g.

This says that if the energy generation is high enough, convection must take over the transport.

(**d**) Finally, use the expressions in your text (or the notes) for energy generation by the pp-chain:

$$\epsilon_{pp} = 2.4 \times 10^4 \, \rho X^2 T_9^{-2/3} e^{-3.38 T_9^{-1/3}} \text{ erg g}^{-1} \text{ s}^1$$

to argue that the Sun has a radiative core. For the Sun, $T_c \sim 1.5 \times 10^7$ K and $\rho_c \sim 150$ g cm^{-3}.

3. *He burning.* (based on HKT 6.6; 20 pts.) Consider the reaction sequence of 3-α, followed by $^{12}\mathrm{C}(\alpha,\gamma)^{16}\mathrm{O}$ and $^{16}\mathrm{O}(\alpha,\gamma)^{20}\mathrm{Ne}$. The set of ODEs that governs the evolution of the number density of these species is:

$$\begin{aligned}
\frac{dn_4}{dt} &= -\frac{3n_4^3}{6}\langle 3\alpha\rangle - n_4 n_{12}\langle \alpha 12\rangle - n_4 n_{16}\langle \alpha 16\rangle \\
\frac{dn_{12}}{dt} &= \frac{n_4^3}{6}\langle 3\alpha\rangle - n_4 n_{12}\langle \alpha 12\rangle \\
\frac{dn_{16}}{dt} &= n_4 n_{12}\langle \alpha 12\rangle - n_4 n_{16}\langle \alpha 16\rangle \\
\frac{dn_{20}}{dt} &= n_4 n_{16}\langle \alpha 16\rangle
\end{aligned}$$

where

- $\langle 3\alpha\rangle = \langle \sigma v\rangle$ for the 3-α reaction. We can find this expressed as $N_A^2\langle\sigma v\rangle_{\alpha\alpha\alpha}$:

$$N_A^2\langle\sigma v\rangle_{\alpha\alpha\alpha} = \frac{2.79\times 10^{-8}}{T_9^3}\exp(-4.4027/T_9) \tag{1}$$

- $\langle \alpha 12\rangle = \langle \sigma v\rangle$ for the $^{12}\mathrm{C}(\alpha,\gamma)^{16}\mathrm{O}$ reaction. We find this expressed as $N_A\langle\sigma v\rangle_{C\alpha}$:

$$\begin{aligned}
N_A\langle\sigma v\rangle_{C\alpha} = \quad & \frac{1.04\times 10^8}{T_9^2(1.0+0.0489/T_9^{2/3})^2}\exp(-32.120/T_9^{1/3} - (T_9/3.496)^2) + \\
& \frac{1.76\times 10^8}{T_9^2(1.0+0.2654/T_9^{2/3})^2}\exp(-32.120/T_9^{1/3}) + \\
& \frac{1.25\times 10^3}{T_9^{3/2}}\exp(-27.499/T_9) + 1.43\times 10^{-2} T_9^5 \exp(-15.541/T_9)
\end{aligned} \tag{2}$$

- $\langle \alpha 16\rangle = \langle \sigma\, v\rangle$ for the $^{16}\mathrm{O}(\alpha,\gamma)^{20}\mathrm{Ne}$ reaction. We find this expressed as $N_A\langle\sigma v\rangle_{O\alpha}$:

$$\begin{aligned}
N_A\langle\sigma v\rangle_{O\alpha} = \quad & \frac{9.37\times 10^9}{T_9^{2/3}}\exp(-39.757/T_9^{1/3} - (T_9/1.586)^2) + \\
& \frac{6.21\times 10^1}{T_9^{3/2}}\exp(-10.297/T_9) + \\
& \frac{5.38\times 10^2}{T_9^{3/2}}\exp(-12.226/T_9) + \\
& 1.30\times 10^1 T_9^2 \exp(-20.093/T_9)
\end{aligned} \tag{3}$$

where

$$T_9 = \frac{T}{10^9\ \mathrm{K}} \tag{4}$$

These expressions come from Caughlin & Fowler (1988) and are in CGS units. Other, much more recent sources exist, so you can look up some updated rates if you like and use them instead.

(**a**) Convert this system of ODEs to one for the molar fractions, Y_k.

(**b**) Integrate this system numerically using a choice of $\rho = 10^5$ g cm^{-3} and $T = 3 \times 10^8$ K.

Note: The timescale for a significant change in $Y(^4\mathrm{He})$ can be estimated by looking at the 3-α rate. That should be used to set the stopping time for your integration.

Depending on your choice of integrator, you will need the Jacobian of this system. You can compute it analytically or numerically via simple finite-differences (some integration packages will do this for you).

(**c**) Make a plot of the change in each *mass fraction* with time.

(**d**) In a paper exploring nucleosynthesis in massive stars, Weaver & Woosley (1993) found that they better matched the observed relative abundances of heavy elements if they multiplied the $^{12}\mathrm{C}(\alpha, \gamma)^{16}\mathrm{O}$ rate by 1.7.

Make this change and rerun your calculation and describe what difference (if any you see).

Feel free to play around with the temperature to see if any other temperature gives different behaviors.

(**e**) Compute the energy released from your burn. You will need the mass excesses for the nuclei in your network, which you can find in the AME 2020 atomic mass evaluation.

You can do the integration in whatever language you wish, but please provide the code.

Some notes on integrating stiff networks and sample python code can be found at: `https://zingale.github.io/comp_astro_tutorial/reaction_networks/stiff-ODEs.html`.