PHY 521: Stars

Homework #4

Due: 2025-10-09 (by the end of the scheduled class period)

Be sure to show all work. Assignment is turned in via Brightspace assignment upload. You should make a single, legible PDF with your solutions. You must show your work and include units with all numbers, all throughout each problem.

While you are free to discuss problems with your classmates, your work must be your own. Any use of AI/LLM must be disclosed. All calculations (including any code you may write) and writing up of the problem solution must be your own work.

Unless otherwise noted, all problems carry equal weight.

1. *Entropy = buoyancy.* (based on Shu) Consider displacing a bubble of fluid upwards in a stratified atmosphere. Show that the condition for convective instability can be expressed as $ds > 0$ in the direction toward the center of the star. That is high entropy material beneath low entropy material is convectively unstable.

Hint: you can express ρ as $\rho(s, P)$, and expand for both the background density change and the density change in the bubble, for some small $d\rho$. Assume that the bubble initially has the same thermodynamic state as the ambient background.

Hint: You can use Maxwell's relations to relate the different derivatives that arise, e.g.,

$$\left.\frac{\partial(1/\rho)}{\partial s}\right|_P = \left.\frac{\partial T}{\partial P}\right|_s \tag{1}$$

and keep in mind that all materials we are dealing with have $\partial T/\partial P|_s > 0$—their temperature increases under adiabatic compression.

2. *Composition gradients.* Consider the criterion for convective instability. In class, we displaced a bubble upwards and found the new density, assuming adiabatic expansion in a medium with a uniform composition. Now let's consider a composition gradient. We'll express our equation of state as:

$$\rho = \rho(P, T, \mu) \tag{2}$$

where μ is the mean molecular weight, which is a function of height. We will express the differential of density as:

$$\frac{d\rho}{\rho} = \underbrace{\left.\left(\frac{\partial \log \rho}{\partial \log P}\right)\right|_{T,\mu}}_{\equiv \alpha} \frac{dP}{P} + \underbrace{\left.\left(\frac{\partial \log \rho}{\partial \log T}\right)\right|_{P,\mu}}_{\equiv -\delta} \frac{dT}{T} + \underbrace{\left.\left(\frac{\partial \log \rho}{\partial \log \mu}\right)\right|_{P,T}}_{\equiv \phi} \frac{d\mu}{\mu} \tag{3}$$

Note that this is basically expressing our EOS as a powerlaw:

$$\rho = \rho_0 P^\alpha T^{-\delta} \mu^\phi \tag{4}$$

and we see that for an ideal gas, $\alpha = \delta = \phi = 1$. Take μ inside the bubble as constant (there are no reactions), but μ is stratified in the background state.

(a) Find the condition for convective instability, and express it as:

$$\nabla > \nabla_{\rm ad} + f(\alpha, \delta, \phi) \nabla_\mu \tag{5}$$

where $\nabla_\mu \equiv (d \log \mu / d \log P)$ of the surroundings. Find the functional form of f.

This is called the *Ledoux criterion* for convection. The gradient of mean molecular weight can stabilize the star against convection. We call the righthand side the Ledoux gradient, ∇_L.

With a composition gradient, the condition for convection then becomes $\nabla > \nabla_L$.

(**b**) Now consider a mixture of an ideal gas and radiation. The pressure in this case is:

$$P = \frac{\rho k T}{\mu m_u} + \frac{1}{3} a T^4 \tag{6}$$

Show that for this mixture, we can write the above Ledoux gradient, ∇_L in terms of the adiabatic ∇_{ad} as:

$$\nabla_L = \nabla_{\text{ad}} + g(\beta) \nabla_\mu \tag{7}$$

where β is the ratio of the ideal gas pressure to the total pressure. Find the function $g(\beta)$ for this case.